Review

Internalizing Identities

**TASK:** Is the LHS equal to the RHS? Why or why not?

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

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\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B \cap C|
\]

**TASK:** Suppose we want to pick \(n\) donuts, and there are \(m\) different flavors of donuts from which we can pick. How many different ways can we pick our \(n\) donuts, and why?
Sort

Sort the following in order from largest to smallest.

i. The number of permutations of all the letters in the alphabet.

ii. The number of subsets of size 6 of the letters of the English alphabet (one such subset is \{a, b, c, d, e, f\}).

iii. The number of 6 letter words made of unique letters (one such 6-letter word is “abcdef”).

iv. The number of (any sized) subsets of the set of the letters in the English alphabet.

v. The number of subsets of size 20 of the letters of the alphabet.

True or False

Suppose we have a function \( f \) from \( X \) to \( Y \). We have a word describing the member to which a given \( x \) maps - namely, the *image* of \( x \). But suppose we have a \( y \in Y \), and we want to know all of the \( x \in X \) that map to \( y \). We actually have a word for this concept too.

We call the set \( \{ x \mid f(x) = y \} \) the *preimage* of \( y \in Y \). That is, the preimage of \( y \) is all of the elements in \( X \) that map to \( y \).

**TASK:** Determine whether the statements are true or false. If a statement is false, modify it to make it true.

**Hint:** Translate the statements we give into pictures.

a. Consider \( f : X \to Y \) such that \(|X| < |Y|\). Then, there exists a \( y \in Y \) such that the preimage of \( Y \) is empty.
b. If $|X| \geq |Y|$, then $f : X \to Y$ cannot be injective.

c. If $|X| > |Y|$, then for all $x_1 \in X$, there exists $x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

d. For all $f : X \to Y$ such that $|X| \geq |Y|$, there exists a function where the preimage of each $y \in Y$ has the same cardinality.

e. Consider $f : X \to Y$, where $|X| = |Y|p + r$, where $p$ and $r$ are integers, and $0 \leq r < |Y|$. Then, there exists an $y \in Y$ such that the cardinality of the preimage of $y$ is at least $p + r$.

f. Suppose the domain, $X$, of $f$ has cardinality $n$, and the codomain, $Y$, has cardinality $k$. Then, there exists a $y \in Y$ such that the cardinality of the preimage of $y$ is exactly $\lceil \frac{n}{k} \rceil$.

The Pigeonhole Principle

The Pigeonhole Principle is a statement about functions. It tells us that if we have a function, $f : |X| \to |Y|$, such that the cardinality of $X$ is $n$ and the cardinality of $Y$ is $k$, then there is some $y \in Y$ such that the preimage of $y$ is greater than or equal to $\lceil \frac{n}{k} \rceil$.

The Pigeonhole Principle gets its power in that we can use functions to think about many real-world situations. For example, we can use a function to think about dividing up $n$ people into $k$ groups. Namely, we can use a function that has $n$ people
in the domain and \( k \) numbers in the codomain, and our function maps each person to their group number.

The pigeonhole principle can help us give a meaningful lower bound on the biggest group. It can tell us other things too: for example, how many people we need in our domain (i.e. what \( n \) needs to be) such that we know there is some group with at least \( z \) people in it (where \( z \) is whatever natural number we want).

**TASK:** Solve the following problems.

**Hint:** Convert the problem into a set up into that involves a function, and then use the Pigeonhole Principle to answer the question.

**Problem 1**

There are infinite red, blue, and yellow socks in a drawer. What is the minimum number of socks we should we pull out to be ensured we have a pair?

**Problem 2**

There are \( n \) people at the ice cream social. Throughout the night they have a series of dance partners.

i. The minimum number of dance partners someone can have is 0. What is the maximum number of dance partners one can have?

ii. Prove that 2 people will have the same number of dance partners by the end of the night.
Problem 3

Suppose $S$ is a set of $n + 1$ integers. Prove that there exist distinct $a, b \in S$ such that $a - b$ is a multiple of $n$.

Problem 4

Given any 5 points inside a square with side length 2, there is always a pair whose distance apart is at most square root of 2.