A Wrap Up to Set Theory

Before we say goodbye to set theory for good, there is an important question to consider. Before we can get to this question though, we’ll need to go through a few things.

First, we say a set $A$ is well-defined when there exists no $x$ such that $x$ must both be in $A$ and not be in $A$. In other words, our definition of the set $A$ has to be non-contradictory; it cannot call for some $x$ to be both in and not in $A$.

Second, a predicate $p(x)$ is a function that takes in an argument, $x$, and evaluates to either true or false. For example, $p(x)$ could be “$x$ is red”. If $x$ were, in fact, red, $p(x)$ would evaluate to true. If $x$ were not red, $p(x)$ would evaluate to false.

We are now ready for our important question. Suppose we write $\{x \mid p(x)\}$. Is $\{x \mid p(x)\}$ necessarily a well-defined set?

Perhaps at first glance, this question seems to have an obvious answer, and that answer is yes. We’ve taught you that $\{x \mid p(x)\}$ is just set builder notation for the set containing all things $x$ that satisfy $p(x)$.

But with a bit more thought, there are some things that are surprising. For example, let’s say $p(x)$ were “$x$ exists in the world”. Then $A = \{x \mid p(x)\}$ would be the set of all things in the world. Interestingly, if $A$ really is a well-defined set, then it would have to contain itself. That is, $A$ would have to contain $A$. Why? Well, the set of all things that exist in the world, well, exists in the world itself!

Before the 20th century, most mathematicians were convinced that for any predicate, $\{x \mid p(x)\}$ described a well-defined set. However, at the dawn of the 20th century, a mathematician named Bertrand Russell came along and posed the following question.
Russell’s question

Consider $S$, where $S = \{X \mid X$ is a set, and $X$ is not a member of $X\}$. For example, $A = \{1, A\}$ would not be in $S$. However, $C = \{1, 2\}$ would be in $S$.

We want to consider the question: is $S$ a member of $S$? That is, is $S$ a member of itself? Let’s consider both possibilities.

a. **Possibility 1:** Suppose $S$ is a member of $S$. From the definition of $S$, what can we conclude?

b. **Possibility 2:** Suppose $S$ is not a member of $S$. From the definition of $S$, what can we conclude?

c. Is $S$ well-defined? Recall what it means for a set to be well-defined: we cannot have an $x$ such that $x$ must both be in and not in $S$.

d. Now, knowing what we know, is it really the case that for any predicate $p(x)$, $
\{x \mid p(x)\}$ is a well-defined set?

e. What does this all mean?!?! Discuss with your neighbors.
Hello Induction

Why does induction work?

Let’s consider an infinite ladder (the best kind of ladder). Suppose we can prove to you both of the following things:

a. You can get to the 1st step of the ladder.
b. If you can get to the kth step of the ladder, then you can get to step k + 1.

★ TASK ★ Why is it the case that for all n ≥ 1, you can get to the nth step of the ladder? Discuss with your neighbors.

Why are we talking about climbing infinite ladders? Well, it turns out this is a good way to think about how induction works.

The base case says that we can reach the first step of the ladder.

The inductive hypothesis says that we can get to the kth step of the ladder.

The inductive step says that if we can get to the kth step of the ladder, then we can get to step k + 1.

Therefore, once we get to step 1, we can get to step 2. Once we get to step 2, we can get to step 3. And so on for all steps of the infinite ladder.

Induction Template

We will now review the template for an inductive proof.

For example, say we are trying to prove that \( \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \) is true for all \( n \in \mathbb{N} \).

1. Define the predicate \( P(n) \). Recall that a predicate is a function that takes in an argument, \( n \), and evaluates to true or false.

   Let \( P(n) \) be the predicate that \( \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \).

\[ \text{If you’re not familiar with this notation, check out } \textcolor{blue}{https://www.mathsisfun.com/algebra/sigma-notation.html} \]
2. Make the assertion that for all \( n \geq a \), where \( a \) is the smallest value we are considering, \( P(n) \) holds.

For all \( n \geq 0 \), \( P(n) \) holds.

3. Show that the base case is true.

We will first show \( P(0) \) is true. \( \sum_{i=0}^{0} i = 0 \) and \( \frac{0(0+1)}{2} = 0 \) so they are equal as needed.

4. State the inductive hypothesis. If you are using standard induction then you will assume \( P(k) \) is true for some integer \( k \geq a \), where \( a \) is your base case value. If you are using strong induction, then you will assume \( P(i) \) is true for all \( i \leq k \). Sometimes, you may need multiple base cases, and you’ll want \( k \) to be greater than or equal to the biggest of them. If you do need multiple base cases, you’ll usually detect this when writing your inductive step (so you’ll need to go back and add more base cases!). In this example though, it turns out we only need one.

Assume \( P(k) \) is true for some integer \( k \geq 0 \).

5. Show that \( P(k + 1) \) is true given the inductive hypothesis.

We will now show that \( \sum_{i=0}^{k+1} i = \frac{(k+1)((k+1)+1)}{2} \).

We know that \( \sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k + 1) \).

By our inductive hypothesis \( \sum_{i=0}^{k} i = \frac{k(k+1)}{2} \).

Therefore

\[
\sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k + 1) = \frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k + 1)((k+1)+1)}{2}
\]

as needed.

6. State that because the base case, \( P(a) \) holds, and because \( P(k) \implies P(k + 1) \), we have that for all \( n \geq a \), \( P(n) \) holds.

Because the base case, \( P(0) \) holds, and because \( P(k) \implies P(k + 1) \), we have that for all \( n \geq 0 \), \( P(n) \) holds.
Warm-up

Prove by induction that for all $n \geq 2$,

\[
(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})... (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}
\]

(!!!) Checkpoint - Call a TA over
**Strong Induction**

Consider a candy bar with \( n \) squares in a row. Suppose we want to break this candy bar up into \( n \) squares. How many breaks should we perform?

![Figure 1: A Delicious Candy Bar](image)

**Claim:** For all \( n \geq 1 \), any sequence of \( n - 1 \) breaks will reduce a candy bar of \( n \) squares into single squares.

⋆ **TASK** ⋆ Prove this claim by induction (Hint: you’ll want to use *strong induction*).
You are climbing a stair case, and you are able to step either one stair at a time or two stairs at a time. Show that the number of unique ways to climb to the $n^{th}$ stair is equal to the $n + 1^{th}$ term in the Fibonacci sequence. Let the $i^{th}$ term, denoted as $F(i)$, of the Fibonacci sequence be defined as follows:

\[
F(1) = 1 \\
F(2) = 1 \\
F(n) = F(n - 1) + F(n - 2)
\]

Hence, the sequence looks like: 1, 1, 2, 3, 5...

(!!!) Checkpoint - Call a TA over
Induction Challenges

**Challenge 1:** We say that an infinite set $S$ is countable if there exists a bijection from $\mathbb{N}$ to $S$. You may assume that $\mathbb{N} \times \mathbb{N}$ is countable (i.e. you do not need to show that the base case in fact holds). Now, prove by induction that $\mathbb{N}^k$ (i.e. $\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}$, $k$ times) is countable for all $k \geq 2$. 
SUPER Challenge 2: $n$ (where $n$ is 2 or greater) dragons are sitting in a circle so that every dragon can see every other dragon.

Every dragon has green eyes. However, no dragon knows its own eye color. Additionally, the dragons cannot talk, so they cannot inform each other of the fact that they have green eyes.

On day 1, Professor Klivans comes and tells the circle of dragons that at least one of them has green eyes.

On the day a dragon realizes it has green eyes, it turns into a human that night.

Prove that on the $n$th night, the $n$ dragons will turn into humans.