Midterm 1 Review
Due: February 24/25, 2016

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Prove the following equality using the element method: \( X \cap Y = (X \cup Y) \setminus (X^c \cup Y^c) \)

Solution

Suppose \( x \in X \cap Y \). Then \( x \in X \) and \( x \in Y \). So \( x \in X \cup Y \). Also \( x \notin X^c \) and \( x \notin Y^c \). So \( x \notin (X^c \cup Y^c) \). So \( x \in (X \cup Y) \setminus (X^c \cup Y^c) \). So \( x \in (X \cup Y) \setminus (X^c \cup Y^c) \).

Thus, \( X \cap Y \) is a subset of \( (X \cup Y) \setminus (X^c \cup Y^c) \).

Suppose \( x \in (X \cup Y) \setminus (X^c \cup Y^c) \). Then \( x \in X \cup Y \) and \( x \notin (X^c \cup Y^c) \). So \( x \notin either X^c \) or \( Y^c \). So \( x \in X \) and \( Y \). So \( x \in X \cap Y \). Thus, \( (X \cup Y) \setminus (X \cap Y) \) is a subset of \( X \cap Y \).

Therefore, \( X \cap Y = (X \cup Y) \setminus (X^c \cup Y^c) \).

Problem 2

Consider two equivalence relations, \( P \) and \( Q \), on a set \( S \).

a. Is \( R_1 = P \cap Q \) an equivalence relation on \( S \)? Prove your answer. If \( R_1 \) is an equivalence relation on \( S \), what can we say about the number of equivalence classes that \( R_1 \) has, relative to \( P \) and \( Q \)?

b. Is \( R_2 = P \cup Q \) an equivalence relation on \( S \)? Prove your answer. If \( R_2 \) is an equivalence relation on \( S \), what can we say about the number of equivalence classes that \( R_2 \) has, relative to \( P \) and \( Q \)?
Solution

a. Yes, \( P \cap Q \) is an equivalence relation.

- Reflexive: \( \forall a \in S, aPa, aQa \), so \( aR_1a \).
- Symmetric: \( aR_1b \) if and only if \( aPb \) and \( aQb \), so by symmetry of \( P \) and \( Q \) we know \( bPa, bQa \). Therefore, \( bR_1a: \ aR_1b \Rightarrow bR_1a \).
- Transitive: By similar logic, \( aR_1b \) and \( bR_1c \) if and only if \( a \) and \( b/c \) and \( d \) are related by both \( P \) and \( Q \). By transitivity of \( P \) and \( Q \), then, we know independently that \( aPc \) and \( aQc \), which implies that \( aR_1c \).

We know that \( R_1 \) must have at least as many equivalence classes as the maximum of \( P \) and \( Q \), since \( R_1 \subseteq P \) and \( R_1 \subseteq Q \). This is because within an equivalence class, all elements are "completely connected" to one another, as necessitated by the properties of reflexivity, symmetry, and transitivity. Since \( R_1 \) is the intersection of \( P \) and \( Q \), an equivalence class in \( R_1 \) will result only when all the elements in this equivalence class are all "completely connected" to one another in both \( P \) and \( Q \). Thus, equivalence classes in \( R_1 \) result from intersections of equivalence classes of \( P \) and \( Q \). However, there can be no fewer intersections of equivalence classes than there are equivalence classes (in one of \( P \) or \( Q \)) to intersect. In other words, if there are \( n \) equivalence classes in \( P \), there will be at least \( n \) intersections of equivalence classes when you consider the equivalence classes of \( Q \). Hence, \( R_1 \) will have no fewer than the maximum of the number of \( P \)'s classes and the number of \( Q \)'s classes.

b. \( R_2 \) is not an equivalence relation. It’s reflexive and symmetric, but not transitive: Consider some \( (a, b) \in P \) and \( (b, c) \in Q \) such that \( (b, c) \notin P \) and \( (a, b) \notin Q \). We know that \( (a, b) \) and \( (b, c) \) are both in \( R_2 \), but since \( (a, c) \) is not in either \( P \) or \( Q \), \( (a, c) \notin R_2 \), violating transitivity for \( R_2 \).

Problem 3

Suppose \( f \) is a bijection between two sets \( A \) and \( B \). Then \( x, y \in A \) gives us \( f(x), f(y) \in B \). Prove that bijections preserve equivalence relations. That is, if \( R \) is an equivalence relation on \( A \), then \( R' \) is an equivalence relation on \( B \), where \( R' \) is defined as the relation such that \( f(x)R'f(y) \) if and only if \( xRy \), where \( x, y \in A \), (therefore \( f(x), f(y) \in B \)).
Solution

a. Suppose \( b \in B \) then \( \exists a \) such that \( f(a) = b \). Since \( R \) is an equivalence relation on \( A \), \( aRa \) which means \( f(a)R'f(a) \) which means \( bR'b \). Since \( b \) was arbitrarily chosen, \( bR'b \) for every \( b \in B \). \((R' \text{ is reflexive})\)

b. Given arbitrary \( b_1 \) and \( b_2 \) such that \( b_1R'b_2 \), \( \exists a_1, a_2 \in A \) such that \( b_1 = f(a_1) \) and \( b_2 = f(a_2) \). This means \( a_1Ra_2 \) which in turn implies that \( a_2Ra_1 \) since \( R \) is symmetric. Therefore \( b_2R'b_1 \). \((R' \text{ is symmetric})\)

c. Now suppose \( b_1, b_2, b_3 \) \( \in B \) such that \( b_1R'b_2 \) and \( b_2R'b_3 \). \( \exists a_1, a_2, a_3 \) with \( b_1 = f(a_1) \), \( b_2 = f(a_2) \) and \( b_3 = f(a_3) \) and \( a_1Ra_2 \), \( a_2Ra_3 \). Since \( R \) is an equivalence relation, \( a_1Ra_3 \). Therefore \( b_1R'a_3 \). \((R' \text{ is transitive})\)

Therefore \( R' \) is also an equivalence relation such that if \( (x, y), (y, x) \in R \), then \( (f(x), f(y)), (f(y), f(x)) \in R' \).

Problem 4

Fry is building a new hallway in the Planet Express, and he wants to know how to tile it. He knows that the hallway will be two units wide, but doesn’t know how long it will be. Your task is to find out how many ways to tile the hallway, using only \( 2 \times 2 \), \( 2 \times 1 \), and \( 1 \times 2 \) tiles, pictured below.

![Tile options](image)

Prove by induction that the number of ways to tile a \( 2 \times n \) hallways is \( \frac{2^{n+1} + (-1)^n}{3} \).

Solution

Proof by induction.

Base case, \( n = 1 \). It is obvious that a \( 2 \times 1 \) hallway can only be tiled in 1 way: with a single \( 2 \times 1 \) block. \( \frac{2^{1+1} + (-1)^1}{3} = \frac{3}{3} = 1 \).

Base case, \( n = 2 \). A \( 2 \times 2 \) hallway can be tiled in 3 ways: two \( 2 \times 1 \) blocks, two \( 1 \times 2 \) blocks, or one \( 2 \times 2 \) block. \( \frac{2^{2+1} + (-1)^2}{3} = \frac{9}{3} = 3 \).
Inductive step. Claim: $T(n) = \frac{2^{n+1} + (-1)^n}{3}$.

Inductive hypothesis: $T(k) = \frac{2^{k+1} + (-1)^k}{3}$ for $0 < k < n, k \in \mathbb{Z}$.

To go from a $2 \times (n - 1)$ board to a $2 \times n$ board, we add one $2 \times 1$ space. This space can be filled with a $2 \times 1$ space, and the remaining $2 \times (n - 1)$ space can be arranged in $T(n - 1)$ ways. We can also consider the $2 \times 1$ space immediately adjacent with the new $2 \times 1$ space separately from the remaining $2 \times (n - 2)$ space. This $2 \times 2$ space can be filled in three ways: two $2 \times 1$ tiles, two $1 \times 2$ tiles, or one $2 \times 2$ tile. However, filling the space with two vertical tiles is entirely redundant relative to some of the tiling schemes counted by $T(n - 1)$ (specifically, the ones in which there is at least one vertical tile), so we only count $2 \cdot T(n - 2)$.

This brings us to the general formula, $T(n) = T(n - 1) + 2T(n - 2)$.

If $n$ is odd (implying $n - 1$ is even and $n - 2$ is odd):

$$T(n) = T(n - 1) + 2T(n - 2)$$
$$= \frac{2^n + (-1)^{n-1}}{3} + 2 \frac{2^{n-1} + (-1)^{n-2}}{3}$$
$$= \frac{2^n + 1 + 2^n - 2}{3}$$
$$= \frac{2^{n+1} - 1}{3}$$

If $n$ is even (implying $n - 1$ is odd and $n - 2$ is even):

$$T(n) = T(n - 1) + 2T(n - 2)$$
$$= \frac{2^n + (-1)^{n-1}}{3} + 2 \frac{2^{n-1} + (-1)^{n-2}}{3}$$
$$= \frac{2^n - 1 + 2^n + 2}{3}$$
$$= \frac{2^{n+1} + 1}{3}$$

so our claim holds.

Problem 5

Michael is counting up the total cent value of the Employee of the Month award and realizes that it is possible to make any positive multiple of 5 cents except for 5 and
15 using only quarters and dimes. Prove that he is correct (for once) using strong induction.

**Solution**

Let $P(n)$ be the statement that you can make $5n$ cents using only quarters and dimes. We must prove that $P(n)$ is true for all positive integers $n$ other than 1 and 3.

**Base cases:** there are three base cases.

- $P(2)$: we can make 10 cents using 1 dime.
- $P(4)$: we can make 20 cents using 2 dimes.
- $P(5)$: we can make 25 cents using 1 quarter.

**Inductive step:** assume $P(k)$ is true for all integers $5 \leq i \leq k$ for some $k \geq 5$. Now we must prove $P(k+1)$.

For all $n$, if $P(n)$ is true, there exist non-negative integers $a, b$ such that $n = 10a + 25b$. $P(k-1)$ is true by our inductive assumption, so let $5(k-1) = 10a + 25b$.

$$5(k + 1) = 5(k - 1) + 10$$
$$= 10a + 25b + 10$$
$$= 10(a + 1) + 25b$$

$a + 1$ is an integer since $a$ is, so there exist integers $r, s$ such that $5(k + 1) = 10r + 25s$. Thus, $P(k+1)$ is true.

**Conclusion:** Since $P(2)$, $P(4)$, and $P(5)$ are true, and $P(i)$ for all integers $5 \leq i \leq k$ for some $k \geq 5$ implies $P(k+1)$, $P(n)$ is true for all positive integers $n$ other than 1 and 3.