Final Exam Review

Due: May 17-19, 2016

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Pam, realizing that the office is a mess, recreates the Chore Wheel and labels it with the numbers 1 through 40, though not necessarily in sequential order. Each number indicates the amount of time that an unlucky soul will have to spend cleaning up the office after work.

a. Prove that, no matter what ordering of numbers Pam labels the Chore Wheel with, there will be at least one sequence of 5 consecutive slots on the wheel with a sum greater than 100.

b. To whip the office slackers into shape, Pam increases the amount of work demanded by the Chore Wheel. Instead of just spinning the wheel and working the number of minutes displayed, each employee at Dunder-Mifflin must spin the wheel twice, add up all of the numbers on the wheel between their first and second spins (wherever they are on the wheel), and spend that many minutes cleaning up. For example, if the order of numbers on the wheel starts with the sequence 1, 7, 23, 14, 30, 6... and Meredith spins the numbers 1 and 30 in that order, her share of the work is $1 + 7 + 23 + 14 + 30 = 75$ minutes.

Prove that at least one of the possible work assignments are divisible by 40.

Problem 2

Jim is angling for a career change, and is training to become a meteorologist. He decides to start monitoring the weather. After several months of staring out the window, he notices some interesting things:

1. In Jim’s neighborhood, it is always either Sunny, or Very Sunny.
2. On any given Monday, there is a 30% chance that it will be Sunny and a 70% chance that it will be Very Sunny.

3. Tuesday’s weather can be predicted based on Monday’s weather. If it was Sunny on Monday, then there is an 80% chance that it will be Very Sunny on Tuesday. If it was Very Sunny on Monday, there is a 60% chance that it will be Very Sunny on Tuesday.

Use this information to answer the following questions:

a. What is the probability that Tuesday’s weather will be Very Sunny?

b. If Tuesday’s weather is Sunny, what is the probability that Monday was also Sunny?

c. If Wednesday’s weather was predictable from Tuesday’s weather in the same way (with the same probabilities) that Tuesday’s weather can be predicted from Monday’s, what is the probability that it is Very Sunny on Wednesday?

Problem 3

Michael is trying to brush up on his math skills. He is given the following equation:

\[ x_1 + x_2 + x_3 + x_4 = 100 \]

where each \( x_i \) must be non-negative, but is not feeling confident that he can figure out a solution. To boost his confidence, Pam decides to count the number of possible solutions to the problem.

a. Count the number of solutions to this equation.

b. Now suppose we require a solution with \( x_1 \) and \( x_4 \) strictly positive. Count the number of solutions under this new constraint.

c. More generally, suppose we require a solution where \( x_i \geq a_i \), where \( a_i \) is a fixed constant nonnegative integer and \( \sum_{i=1}^{4} a_i \leq 100 \). Again, count the number of solutions under this new constraint.

Problem 4

Prove that by connecting any number of connected graphs that have connected Euler circuits together the resulting graph will also have an Euler circuit. In this problem
we define connecting as merging any two vertices together into a single new vertex, as depicted in the image below:

Problem 5

Consider paths from the bottom left corner to the top right corner of a rectangular grid. The picture below shows a $4 \times 10$ grid with one path highlighted.

a. Count the number of shortest paths of an $n \times k$ grid from the bottom left corner to the top right corner.

b. Prove that if $|k - n| > 1$ then there will be at least two consecutive up steps or two consecutive right steps in any shortest path.