Problem Session 2

Due: Wednesday, February 10, 2016, and Sunday, February 14, 2016

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

For the following relations, denoted $R$, give the inverse relation.

a. $R = \{(1, 1), (1, 2), (2, 3), (3, 5), (5, 8)\}$

b. $R = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$

c. $(a, b) \in R \iff a \geq b$

d. $(a, b) \in R \iff a = 2^b$

Solution

a. $R^{-1} = \{(1, 1), (2, 1), (3, 2), (5, 3), (8, 5)\}$

b. $R^{-1} = \{(1, 1), (4, 2), (9, 3), (16, 4), (25, 5)\}$

c. $(a, b) \in R^{-1} \iff a \leq b$

d. $(a, b) \in R^{-1} \iff b = 2^a \iff b > 0 \text{ and } a = \log_2(b)$

Problem 2

Define $S$ as the set of 0/1 bit strings and the function $f : S \to \mathbb{Z}$ as follows: $f(x) =$ the number of 1’s in $x$.

a. Show that $f$ is not injective (one-to-one).
b. Give a new domain $A \subseteq S$ such that $f : A \rightarrow \mathbb{Z}$ is injective, and prove its injectivity. ($f(x)$ is still the number of 1's in $x$.)

c. Show that your new $f$ is not surjective (onto).

d. Give a new codomain $B \subseteq \mathbb{Z}$ such that $f : A \rightarrow B$ is surjective, and prove its surjectivity. (Again, $f(x)$ is still the number of 1's in $x$.)

e. What can you conclude about cardinalities of your $A$ and $B$? Why?

**Solution**

a. $f$ is not injective, since multiple 0/1 strings can have the same number of 1s in them. As a counterexample, $f(01) = f(10) = 1$, but 01 $\neq$ 10.

b. Define $A$ as the set of strings composed of just 1’s. Now $f : A \rightarrow \mathbb{Z}$ is injective, since if two strings $x$ and $y$ in $A$ have the same number of 1’s, they are the same string, $x = y$.

c. $f$ is not surjective, since a bit string cannot contain a negative number of 1’s. As a counterexample, $-4 \in \mathbb{Z}$ is not mapped to, because no bit string contains -4 1’s.

d. Define $B$ as the set of non-negative integers $\mathbb{N}$. Now $f : A \rightarrow B$ is surjective, since any non-negative integer $n \in \mathbb{N}$ is mapped to by a 1-string $x$ of length $n$.

e. Note that $f$ is still injective, since there are still no two distinct elements in $A$. Thus, $f$ is both injective and surjective, so it is bijective. Thus we know that $|A| = |B|$.

**Problem 3**

For any relation $R$, define $R^{-1} = \{(y, x) \mid (x, y) \in R\}$, i.e. “reverse the pairings.” Define the “divides” relation $D$ on $\mathbb{Z}$ as $(a, b) \in D$ iff $a \mid b$.

For each of the following relations on $\mathbb{Z}$, prove whether it is reflexive, symmetric, and/or transitive.

a. $A = D \cap D^{-1}$

b. $B = D \cup D^{-1}$
Solution

**Reflexive:** Since every integer divides itself, \((x, x) \in D\) for all \(x \in \mathbb{Z}\), which implies that \((x, x) \in D^{-1}\). Therefore, since both \(D\) and \(D^{-1}\) are reflexive, both their intersection and their union are reflexive, and so both \(A\) and \(B\) are reflexive.

**Symmetric:** The only pairs contained in the intersection of \(D\) and \(D^{-1}\) are reflexive pairs, i.e. pairs where for \((x, y) \in D\), \(x = y\). Therefore, \(A\) is symmetric. Since \(D^{-1}\) contains all of the symmetric pairs for the pairs in \(D\), the union of the two relations is symmetric. Therefore \(B\) is symmetric, as well.

**Transitive:** \(B\) is not transitive. Consider the following counterexample:

Consider the pair \((2, 6)\). Since \(2 \mid 6\), \((2, 6) \in D\). Thus \((2, 6) \in B\) by definition of \(B\).

Now consider the pair \((4, 2)\). Since \(2 \mid 4\), \((2, 4) \in D\), so \((4, 2) \in D^{-1}\). Thus \((4, 2) \in B\) by definition of \(B\).

Lastly, consider the pair \((4, 6)\). We see \(4 \nmid 6\), so \((4, 6) \notin D\). Additionally, \(6 \nmid 4\), so \((4, 6) \notin D^{-1}\). Thus, \((4, 6) \notin B\).

So \((4, 2) \in B, (2, 6) \in B\), but \((4, 6) \notin B\). This violates the definition of transitivity, so \(B\) is not transitive.

**Problem 4**

Prove that the number of non-negative integers is the same as the number of perfect squares.

**Solution**

The cardnalities of the set of non-negative integers and the set of perfect squares are equivalent if there exists a bijection between the two, as in there exists some one-to-one function that maps each of the non-negative integers to each of the perfect squares. Let \(f(x) = x^2\) for all \(x \geq 0\).

For every non-negative integer \(x\), there exists exactly one output for \(f(x)\) which lies in the set of perfect squares. Thus this function is injective.

For each of the perfect squares, there exists exactly one *non-negative* integer which maps to that perfect square using this function. For example, the only
non-negative integer input that would make this function output a 25 is the non-negative integer 5. Therefore this function is surjective.

Since this function is both injective and surjective, it describes a bijection between the set of non-negative integers and the set of perfect squares, therefore the number of non-negative integers must be the same as the number of perfect squares.

**Problem 5**

Michael is a very expressive character, so the members of the office try to gauge how he is feeling on any given day of the week so they can anticipate how the rest of the day will go. Everyone in the office thinks Michael’s mood might have something to do with which day of the week it is, but they are not sure exactly how.

Let $R = \{\text{happy, sad, bossy, beligerent}\}$ be the set of Michael’s potential moods. Let $V = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday}\}$ be the set of work days.

For each of the following sets of ordered pairs representing possible ways in which the given week day affects Michael’s mood, state whether the set determines a function from $V$ to $R$.

If a set of ordered pairs does not determine a function, explain why not. If a set of ordered pairs does determine a function, state the range, the inverse image of “happy,” whether or not the function is injective, whether or not the function is surjective, and whether or not the function is bijective.

a. $\{(\text{Monday, happy}), (\text{Tuesday, bossy}), (\text{Thursday, happy})\}$

b. $\{(\text{Monday, sad}), (\text{Tuesday, sad}), (\text{Wednesday, sad}), (\text{Thursday, sad}), (\text{Friday, happy})\}$

c. $\{(\text{Monday, beligerent}), (\text{Tuesday, bossy}), (\text{Wednesday, sad}), (\text{Thursday, bossy}), (\text{Friday, happy})\}$

d. $\{(\text{Monday, happy}), (\text{Wednesday, sad}), (\text{Thursday, bossy}), (\text{Thursday, beligerent})\}$

**Solution**

a. This set does NOT determine a function, because not every input value in $V$ has an output in $R$. For example, “Wednesday” is not paired with an output value.

b. This set determines a function.
• The range is \{happy, sad\}.
• The inverse image of “happy” is \{Friday\}.
• This function is not injective, because e.g. “Monday” and “Tuesday” both map to “sad”.
• This function is not surjective, because e.g. “bossy” is not mapped to.
• Thus, this function is not bijective.

c. This set determines a function.

• The range is \{happy, sad, bossy, beligerent\}.
• The inverse image of “happy” is \{Friday\}.
• This function is not injective, because “Tuesday” and “Thursday” both map to “bossy”.
• This function is surjective because each member of \( R \) has at least one matching value from \( V \), i.e. every member of \( R \) is mapped to.
• Thus, this function is not bijective.

d. This set does not determine a function, because there are two outputs for one input: “Thursday” maps to two distinct values.