Homework 10
Due: Thursday, May 5 at 2:30pm

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Dwight and Mose are sending encrypted messages to each other using dots and dashes. The probability that a particular signal is a dot is 0.4. Due to noisy transmission, a dot becomes a dash with probability $a = 0.25$, while a dash becomes a dot with probability $b = 0.75$. All signals are independent of each other.

Suppose Mose sends a 10-signal message to Dwight. Let $X$ be the number of dots that were actually sent, and $Y$ be the number of dots that were received.

Note: By definition, two random variables $X$ and $Y$ are independent if

$$P(X = k \text{ and } Y = m) = P(X = k)P(Y = m)$$

for all $k, m \in \mathbb{R}$.

a. Find the expected value and variance of $X$ and $Y$.

b. Using Markov’s Inequality or Chebyshev’s Inequality, find an upper bound for the following probabilities on a 10-signal message before transmission.

   i. The probability that more than 5 of signals being sent are dots.

   ii. The probability that less than 2 or more than 6 of the signals sent are dots.

c. We defined specific probabilities of error in transmission as $a$ and $b$ above. What property must any such $a$ and $b$ satisfy for $X$ and $Y$ to be independent? Prove that if your property holds, $X$ and $Y$ are independent.

d. Find the expected value and variance of the “total error” $Z = Y - X$.

e. Given that Dwight received 10 dots, find $P(X \geq 1)$.
Problem 2

Let $K_n = (V_n, E_n)$ denote the complete graph on $n$ vertices.

We define a random graph as follows. $G = (V_n, E)$ is a random graph on $n$ vertices if $G$ is a subgraph of $K_n$ such that $E \subseteq E_n$ is constructed as follows: For each possible edge $e \in E_n$, we include $e$ in $E$ with probability $p$, and exclude $e$ with probability $1 - p$.

a. What is the probability that a random graph $G = (V_n, E)$ on $n$ vertices is the complete graph $K_n$?

For any graph $G = (V, E)$, consider a set $T = \{u, v, w\}$ such that $T \subseteq V$ and $|T| = 3$. We say that $T$ forms a triangle in $G$ if $(u, v), (v, w),$ and $(u, w)$ are all elements of $E$.

b. How many distinct triangles are in $K_n$?

c. Let $T \subseteq V_n$ be a triangle in $K_n$. What is the probability that, for a random graph $G = (V_n, E)$ on $n$ vertices, that $T$ is also a triangle in $G$?

d. What is the expected number of triangles in a random graph on $n$ vertices?

Problem 3

After an unfortunate accident, Michael is trying to plan his first road race, Michael Scott’s Dunder Mifflin Scranton Meredith Palmer Memorial Celebrity Rabies Awareness Pro-Am Fun Run Race for the Cure. The race will start at the Scranton office, travel along roads to locations in Scranton, and finish back at the Scranton office. In order to maximize exposure for his cause, Michael wants his race’s route to travel every road in Scranton, but not use any road twice. All roads are two-way, and all locations are reachable by road from the Scranton office.

We will prove that Michael holds the race if and only if each location in Scranton has an even number of roads connecting to it. Let us model the race on a connected simple graph $G = (V, E)$, where elements of $V$ represent locations and elements of $E$ represent roads.

a. What information from the problem tells us that $G$ is connected? Explain why finding an acceptable route for Michael’s race is equivalent to finding an Eulerian cycle on $G$.

b. Prove that if $G$ has an Eulerian cycle, then every vertex $v \in V$ has even degree.
c. Prove, by induction on the number of edges, that if every vertex \( v \in V \) has even degree, then \( G \) has an Eulerian cycle.

   **Note:** See the Hint in Problem 4, part d.

We conclude that a connected simple graph \( G = (V, E) \) has an Eulerian cycle if and only if every vertex \( v \in V \) has even degree.

For all \( n \geq 1 \), the graph \( \text{Cube}_n \) is defined as follows. The vertex set is \( \{0, 1\}^n \) and, for binary strings \( u \) and \( v \), \( (u, v) \) is an edge of \( \text{Cube}_n \) if and only if \( u \) and \( v \) differ in exactly one position. For example, \( \text{Cube}_1 \) is a single edge, \( \text{Cube}_2 \) is a square, \( \text{Cube}_3 \) is a 3-dimensional cube. (If it aids your understanding, try drawing these small examples.)

For each of the following, be sure to justify your answers.

d. What is the degree of an arbitrary vertex in \( \text{Cube}_n \)?

e. For which values of \( n \) does \( \text{Cube}_n \) have an Eulerian cycle?

f. Let \( n > 2 \), and consider some \( \text{Cube}_n \) that has an Eulerian cycle. Suppose some random edge is removed from \( \text{Cube}_n \). What is the minimum number of edges you’d need to further remove such that the remaining graph has an Eulerian cycle?

   **Note:** You can choose which further edges to remove.

**Problem 4**

The degree of a vertex \( v \) is the number of edges \( e \in E \) such \( v \) is one of the endpoints of \( e \). We denote the degree of vertex \( v \) by \( \deg(v) \).

a. Prove using a counting argument that \( \sum_{v \in V} \deg(v) = 2|E| \).

We say a graph is \( n \)-colorable if we can *properly color* a graph using \( n \) colors. A graph is properly colored if each vertex in the graph is assigned a color such that for all edges \( (u, v) \), \( u \) and \( v \) are assigned *different* colors.

b. For the following graphs, state the minimum number of colors needed to properly color the graph, and provide a coloring using that many colors.

   i. \( G = (V, E) \) where \( V = \{1, 2, 3\} \) and \( E = \{(1, 2), (1, 3), (2, 3)\} \).

   ii. \( G = (V, E) \) where \( V = \{1, 2, 3, 4\} \) and \( E = \{(1, 2), (2, 3), (3, 4)\} \).
A graph is planar if it can be drawn on a plane in such a way that its edges intersect only at vertices. In other words, we can draw the graph on a piece of paper in such a way that no two edges overlap.

The 4-color theorem says that any planar graph can be properly colored using only 4 colors. This theorem is famously difficult to prove... but you can prove the 6-color theorem right now! Let $G = (V, E)$ be a simple, connected, planar graph on at least three vertices. We will be using this graph for the rest of the problem.

c. Start by using your result from part a. to show that the average (mean) degree of vertices in $V$ is strictly less than 6.

**Hint:** You may use without proof that, for any simple, connected, planar graph on at least 3 vertices, $|E| \leq 3|V| - 6$.

d. Show by contradiction that there exists a vertex $v \in V$ such that $\deg(v) \leq 5$.

e. Prove, by induction on the number of vertices, that $G$ is 6-colorable.

**Hint:** You are going to want to build down. In other words, when you are trying to prove that your claim is true for a graph on $n + 1$ vertices, do not start with a graph on $n$ vertices and add one more: it’s very difficult to cover all graphs on $n + 1$ vertices this way. Instead, start with an arbitrary planar graph on $n + 1$ vertices, then find a way to remove one vertex to form a subgraph on $n$ vertices, where you can use your inductive hypothesis.