Homework 9

Due: Wednesday, April 20

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

a. For each of the following pairs of events, identify whether they are independent and justify why or why not.

i. When rolling a fair die:
   - rolling an even number
   - rolling a number greater than three

ii. When flipping a fair coin twice:
   - the first flip is a heads
   - the two flips match

iii. When flipping a fair coin three times:
   - the first coin is a tails
   - there is a run of exactly two heads (i.e. two, but not three, heads are flipped in a row)

Dwight buys a lie detector that is 97% accurate; that is, if someone is lying, it will indicate that with probability .97, and if someone is telling the truth, it will also indicate that with probability .97.

Dwight decides to test it on 30 Dunder Mifflin employees. Knowing that exactly two of them were late to work in the last week, Dwight asks each of them whether they had committed this horrible crime. Assume that the results for the employees are all independent of each other, and that all employees will claim to be innocent.

b. What is the probability that an employee is guilty given that the lie detector says the employee is lying?
c. What is the probability that the lie detector correctly determines the guilt or innocence of all the employees?

**Problem 2**

In 1334, Pope Benedict XII became pope by accident. When the Cardinals held the first ballot to elect a new pope from among their ranks, each voted randomly, thinking he could see how the other Cardinals were leaning. Amazingly, one Cardinal won!\(^1\)

a. Assuming the \( n \) Cardinals cast their votes uniformly at random, what is the probability that one Cardinal receives a majority (greater than 50%) of the votes cast?

b. It is not very sporting to vote for oneself. If each Cardinal votes uniformly at random among the candidates who are not himself, what is the probability that one Cardinal receives a majority?

**Note:** Your answers do not need to be in closed form.

**Problem 3**

Dunder Mifflin organized a weight loss contest, but Michael saw the giant balance and decided to play an *incredibly* fun game instead. There are \( n \) employees, and their weights all happen to be powers of 2: \( 2^5, 2^6, \ldots, 2^{n+4} \). In each round, Michael chooses one of the employees that hasn’t been placed yet, and puts them on either the left or right side of the balance. Michael wins once everyone is standing on the balance, but loses if the right side is ever heavier than the left.

a. How many ways can Michael win the game? (You may leave the answer in expanded form).

b. If Michael chooses an employee uniformly at random in each round, and assigns them to either side uniformly at random, what is the probability that he wins the game?

\(^{1}\)Based on a true story [http://en.wikipedia.org/wiki/Pope_Benedict_XII#Fournier.27s_accession_to_the_Papacy](http://en.wikipedia.org/wiki/Pope_Benedict_XII#Fournier.27s_accession_to_the_Papacy)
Problem 4

Michael Scott has a coupon to buy two pizzas from the local pizza shop, *Pizza by Alfredo*. On each of the pizzas, he can add between 0 and 11 different toppings (inclusive), but cannot get any topping more than once on a single pizza.

Michael has spent all of the past work week trying to figure out how many ways he could order. He has reasoned that, since each pizza can have 11 toppings, there are $2^{11}$ possible ways to create a pizza. That means that there are $(2^{11})^2 = 4194304$ total possible combinations for an order of two pizzas—all at an incredible coupon bargain!

Unfortunately, Michael (having never taken CS22) is incorrect in his calculation.

a. Convince Michael that he is wrong by either giving an example of a pizza combination he counted more than once, or giving a pizza combination he did not account for.

b. How many ways can you really choose the toppings for two pizzas?

c. How many ways can you choose toppings for $n$ pizzas?

d. Suppose that 4 out of the 11 possible toppings are meat toppings. Michael orders one pizza and chooses its toppings uniformly at random out of the total set of possibilities for toppings. What is the probability that Angela, who is a vegetarian, will be able to eat it? (Luckily, Angela would eat any pizza as long as there is no meat on it.)

e. What is the probability that Angela will be able to eat at least one pizza if $n$ pizzas are ordered independently with their toppings chosen as in part (d)?

Problem 5

Suppose that you have two dice. One (die $X$) is a fair die, but the other one (die $L$) is loaded, as shown below:

<table>
<thead>
<tr>
<th>Side</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
</tr>
<tr>
<td>2</td>
<td>.25</td>
</tr>
<tr>
<td>3</td>
<td>.1</td>
</tr>
<tr>
<td>4</td>
<td>.05</td>
</tr>
<tr>
<td>5</td>
<td>.05</td>
</tr>
<tr>
<td>6</td>
<td>.05</td>
</tr>
</tbody>
</table>

a. What is the probability of rolling a sum of seven with these two dice?
b. Suppose you pick up one of the dice at random and roll a 2. What is the probability that you picked up die \(L\)?

You want to use your loaded die to cheat another player out of their money, and decide on the following game. You and the other player will each bet one dollar and roll a die: whoever rolls the lower number wins both dollars (and a tie means the bets are returned).

c. If you use die \(L\) while the other player uses die \(X\), calculate your expected winnings after one round.

The second player has gotten suspicious, and is testing the dice in order to make sure you’re not cheating. While their back is turned, you have the chance to switch the die \(X\) with a second loaded die (die \(Y\)), in order to trick their test.

**Note:** Die \(L\) is not replaced.

d. The second player announces that they will roll the dice an arbitrary number of times, and record the number of times that the roll sums to seven. How should you load die \(Y\) so that the other player is convinced the dice are fair? Give a particular probability distribution.

e. Convinced, the second player wants to play the game for a few more rounds to try to win their money back. Calculate your expected winnings now that the second player is using \(Y\).

Suspicious again, the second player wants to do another test. This time, they will roll the dice \(k\) times each (for arbitrary, large \(k\)), and will be appeased if the two dice roll 2’s approximately the same number of times. You have the chance to switch out die \(Y\) for a new loaded die, \(Z\).

f. You expect that you will be able to play a few rounds of the game after the testing is done. How should you load \(Z\) so that it passes the test, but so that your winnings in the game will be maximized? Give a particular probability distribution.

g. Calculate the expected winnings of a round of the game given the other player uses \(Z\) and you use die \(L\).