Homework 8
Due: Wednesday, April 20

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

a. Let \( S = \{1, 2, \ldots, 2n\} \) for some integer \( n \). Show that for any \( T \subset S \) such that \( |T| = n + 1 \), there are elements \( x, y \in T \) such that \( x \) and \( y \) are relatively prime.

b. A repunit is a number that contains only the number 1 (1, 11, 111, 1111, etc.).

Prove, using the pigeonhole principle, that among the first 50 repunits, at least one of them is divisible by 49.

c. Assume there are 101 dalmatians, each of which has some nonnegative, integer number of spots. Prove that it is possible to choose 11 of them whose total number of spots is divisible by 11.

Problem 2

Jim is planning the latest series of office pranks and has come up with a massive list of potential pranks. However, only some of these are worthy of his time and effort. A good prank is one that is inexpensive to put together, entertaining to execute and watch, and not dangerous.

a. Of Jim’s 155 prank ideas, 52 are dangerous, 17 are dangerous but entertaining, 12 are dangerous but inexpensive, and two pranks are both inexpensive and entertaining, but also dangerous. No prank has none of the three properties. If a total of 80 pranks are inexpensive and only 67 of the 155 pranks are entertaining, how many good pranks does Jim have to work with?

b. Jim wants to use all of his “good” prank ideas over the next work week - 5 days in total.
i. Assume the pranks are all indistinguishable, and assume that Jim cannot repeat a prank. How many ways can he spread out all the good pranks across these five days?

ii. In how many ways can he spread out the pranks across the 5 days if he pulls at least one prank every day?

c. Repeat part (b), assuming that each prank is distinct (that is, reversing the order in which two pranks are pulled, even on the same day, results in a distinct prank schedule).

**Problem 3**

Phyllis is participating in the Office Olympic game of Flonkerton, the national sport of Icelandic paper companies. She wins the first round but loses in the second round. For every round thereafter, the probability that she wins that round is exactly equal to the proportion of rounds that she has already won. For example, since Phyllis won one of the first two rounds, she has a \( \frac{1}{2} \) chance of winning the third round. What is the probability that after 100 rounds of Flonkerton she will have won exactly 50 rounds?

**Note:** There are no ties in Flonkerton, so each round must either be won or lost.

**Problem 4**

a. For \( n > 0 \) and \( k \geq 0 \), let \( f(n) = \binom{n}{k} \) and let \( g(n) = n^k \). Prove that \( f(n) \in O(g(n)) \).

b. Let \( f_1(n) \in O(g_1(n)) \) and \( f_2(n) \in O(g_2(n)) \). Show that
\[
\sum_{k=0}^{n} \binom{n}{k} k!(n-k)! \in O(\max(|g_1(n)|, |g_2(n)|)).
\]

**Problem 5**

Prove by showing both sides count the same thing that
\[
(n+1)(n!) = \sum_{k=0}^{n} \binom{n}{k} k!(n-k)!
\]
for \( n \geq 1 \).

**Hint:** Consider a set of size \( n \) and some fixed \( k \) between 0 and \( n \). What does \( \binom{n}{k} \) indicate in this situation? What does \( k!(n-k)! \) represent?