CSCI 0220 Discrete Structures and Probability  
C. Klivans

Homework 8
Due: Wednesday, April 20

All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

a. Let $S = \{1, 2, \ldots, 2n\}$ for some integer $n$. Show that for any $T \subset S$ such that $|T| = n + 1$, there are elements $x, y \in T$ such that $x$ and $y$ are relatively prime.

b. A repunit is a number that contains only the number 1 ($1, 11, 111, 1111, \text{etc.}$). Prove, using the pigeonhole principle, that among the first 50 repunits, at least one of them is divisible by 49.

c. Assume there are 101 dalmatians, each of which has some nonnegative, integer number of spots. Prove that it is possible to choose 11 of them whose total number of spots is divisible by 11.

Solution

a. First, we will prove that $T$ contains two elements $x, y$ that are adjacent, that is, $|x - y| = 1$.

To do this, first assume that there does not exist two elements $x, y$ that are adjacent. Then for any distinct $x, y$, $|x - y| \geq 2$. But by the pigeonhole principle, any subset of $S$ that satisfies this constraint has at most $\left\lfloor \frac{|S|}{2} \right\rfloor = n$ elements. This contradicts the assumption that $|T| = n + 1$. Thus there is some pair $x, y \in T$ such that $|x - y| = 1$.

Now, we show that $x$ and $y$ are relatively prime. Assume there is some $d$ such that $d|x$ and $d|y$. Then $d|(x - y)$, so $d|1$, so $d = 1$. Therefore $x$ and $y$ are relatively prime.

b. Of the first 50 repunits, at least two of them must have the same remainder when divided by 49 by the pigeonhole principle. This means that their
difference \( n \) must be divisible by 49.

The difference \( n \) is a series of 1’s, followed by a series of 0’s, so \( n \) is the product of a repunit \( m \) and a power of 10. Since 10 shares no common factors with 49, that repunit \( m \) must be divisible by 49.

c. Assign each dalmatian a number equal to the equivalence class of the number of spots it has \((\text{mod } 11)\). You now have 101 dalmatians numbered between 0 and 10 inclusive. Separate the dalmatians into groups based on their assigned number, which gives 11 groups.

Case 1: There is an empty group. In this case, we are separating 101 dalmatians into only 10 groups. Since \( \frac{101}{10} \) gives a number that is bigger than 10, at least one of the groups must contain at least 11 dalmatians. Therefore, you can then choose 11 dalmatians from one group. Adding their spots \((\text{mod } 11)\) gives a result that is congruent to 0 \((\text{mod } 11)\) since \( 11 \times a \ (\text{mod } 11) \equiv 0 \) for any \( a \in \mathbb{Z} \) where \( a \) is the number of whichever group was chosen.

Case 2: There is no empty group. In this case, there is at least one dalmatian in each group. Choose one dalmatian from each group. Those dalmatians’ total number of spots is congruent to 0 \((\text{mod } 11)\) since \( 0 + 1 + 2 + \ldots + 10 = 55 = 5 \times 11 \).

Since in both cases, which are mutually exclusive and which cover all possible cases, you are able to choose 11 dalmatians where the sum of the number of spots on the 11 dalmatians is congruent to 0 \((\text{mod } 11)\), then it is possible to pick 11 dalmatians from 101 dalmatians whose total number of spots is divisible by 11.

Problem 2

Jim is planning the latest series of office pranks and has come up with a massive list of potential pranks. However, only some of these are worthy of his time and effort. A good prank is one that is inexpensive to put together, entertaining to execute and watch, and not dangerous.

a. Of Jim’s 155 prank ideas, 52 are dangerous, 17 are dangerous but entertaining, 12 are dangerous but inexpensive, and two pranks are both inexpensive and entertaining, but also dangerous. No prank has none of the three properties. If a total of 80 pranks are inexpensive and only 67 of the 155 pranks are entertaining, how many good pranks does Jim have to work with?

b. Jim wants to use all of his “good” prank ideas over the next work week - 5 days in total.

i. Assume the pranks are all indistinguishable, and assume that Jim cannot
repeat a prank. How many ways can he spread out all the good pranks across these five days?

ii. In how many ways can he spread out the pranks across the 5 days if he pulls at least one prank every day?

c. Repeat part (b), assuming that each prank is distinct (that is, reversing the order in which two pranks are pulled, even on the same day, results in a distinct prank schedule).

Solution

a. Let $D$, $E$, and $I$ represent the sets of dangerous, entertaining, and inexpensive pranks, respectively. We can solve for the number of good pranks ($(E \cap I) \setminus D$) by applying the inclusion-exclusion principle. $|D \cap E| = 17$, $|D \cap I| = 12$, and $|D \cap E \cap I| = 2$, so we know that $|D \cap E \setminus I| = 15$, $|D \cap I \setminus E| = 10$. From this we can see that $D \setminus E \setminus I$ contains just $|D \setminus E \setminus I| = 52 - 2 - 10 - 15 = 25$ pranks (You can draw a Venn diagram if you’re not convinced of this). This leaves $155 - 25 = 130$ pranks that are entertaining, inexpensive, or both. By inclusion-exclusion, the size of the intersection $|E \cap I|$ is $|E| + |I| - |E \cup I| = 80 + 67 - 130 = 17$. Of these 17 remaining pranks, we know that $|D \cap E \cap I| = 2$ of them are dangerous, which leaves Jim with just 15 good pranks to use.

b. Distributing 15 pranks across 5 days can be done with the stars-and-bars method: For the first part of this problem, the $\binom{n+k-1}{k-1}$ form of stars and bars applies, resulting in $\binom{19}{4} = 3876$ ways, and the second part uses the $\binom{n-1}{k-1}$ form of stars-and-bars for a total of $\binom{14}{4} = 1001$ prank schedules.

c. We can use stars-and-bars, as above, to find the number of ways to distribute the pranks across 5 days. The only difference here is that the ordering of individual pranks matters now. There are 15! permutations of 15 distinct pranks, so there are a total of $15! \times 1001 \approx 1.30898 \times 10^{15}$ and $15! \times 126 \approx 1.64767 \times 10^{14}$ prank schedules for the first and second parts of the problem, respectively.

Problem 3

Phyllis is participating in the Office Olympic game of Flonkerton, the national sport of Icelandic paper companies. She wins the first round but loses in the second round. For every round thereafter, the probability that she wins that round is exactly equal to the proportion of rounds that she has already won. For example, since Phyllis won one of the first two rounds, she has a $\frac{1}{2}$ chance of winning the third round. What
is the probability that after 100 rounds of Flonkerton she will have won exactly 50 rounds?

**Note:** There are no ties in Flonkerton, so each round must either be won or lost.

**Solution**

To look for a pattern, we can examine one possible outcome: if Phyllis continues the pattern of a win followed by a loss, the probability of this specific outcome will be \(\frac{1}{2} \times \frac{1}{3} \times \frac{2}{4} \times \frac{2}{5} \times \frac{3}{6} \times \frac{3}{7} \times \ldots \times \frac{49}{98} \times \frac{49}{99} = \frac{(49!)^2}{99!}\).

It is clear that for any sequence, the denominators of the above fractions will be the same (since the \(n^{th}\) shot will always have a probability \(\frac{k}{n}\) of being a win, where \(k\) is the number of games that has already been won, and \(n\), the denominator, does not depend on how many shots have been made). For the \(j^{th}\) won game, the numerator of its probability is \(j - 1\) since \(j - 1\) games have already been won. Similarly, for the \(j^{th}\) lost game, the numerator of its probability is also \(j - 1\) since \(j - 1\) games have already been lost, and the probability of losing is \(1 - \frac{\text{gamesWon}}{\text{totalGames}} = \frac{\text{gamesWon} + \text{gamesLost}}{\text{totalGames}} - \frac{\text{gamesWon}}{\text{totalGames}} = \frac{\text{gamesLost}}{\text{totalGames}}\). So, since all possible sequences have 50 wins and 50 losses, the order of the numerators will differ by sequence, the product of all the numerators in each sequence will not change. Therefore, since there is a fixed numerator and denominator product for each sequence, each has a probability of \(\frac{(49!)^2}{99!}\) (the same as the probability in our example). The total number of possible sequences is the total number of ways that Phyllis can win 49 of the remaining 98 games, or \(\binom{98}{49}\). Therefore the total probability of getting exactly 50 wins is \(\binom{98}{49} \times \frac{(49!)^2}{99!} = \frac{98!}{49!49!} \times \frac{(49!)^2}{99!} = \frac{98!}{99!} = \frac{1}{99}\).

**Problem 4**

a. For \(n > 0\) and \(k \geq 0\), let \(f(n) = \binom{n}{k}\) and let \(g(n) = n^k\). Prove that \(f(n) \in O(g(n))\).

b. Let \(f_1(n) \in O(g_1(n))\) and \(f_2(n) \in O(g_2(n))\). Show that

\[
f_1(n) + f_2(n) \in O(\max(|g_1(n)|, |g_2(n)|)).
\]
Solution

a. Expanding the term $\binom{n}{k}$ we have:

$$f(n) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{k!}{k!} \prod_{j=1}^{k} (n-k+j)$$

Since $(n-k+j) \leq n$ for $1 \leq j \leq k$, we have

$$\frac{k!}{k!} \prod_{j=1}^{k} (n-k+j) \leq \frac{k!}{k!} = \frac{n^k}{k!}$$

And, finally, since $k! \geq 1$ we have

$$\binom{n}{k} \leq \frac{n^k}{k!} \leq n^k$$

so we have shown that $f(n) \leq g(n)$ and therefore $f(n) \in O(g(n))$ for $n > 0$ and $k \geq 0$.

b. Since we have $f_1(n) \in O(g_1(n))$ we know that $\forall n \geq n_1$ we have $|f_1(n)| \leq c_1|g_1(n)|$ for some $c_1 > 0$. Similarly, since we have $f_2(n) \in O(g_2(n))$ we know that $\forall n \geq n_2$ we have $|f_2(n)| \leq c_2|g_2(n)|$ for some $c_2 > 0$.

We have

$$f_1(n) + f_2(n) \leq |f_1(n)| + |f_2(n)|$$

$$\leq 2(\max(|f_1(n)|, |f_2(n)|))$$

Let $c_* = \max(c_1, c_2)$ and $n_* = \max(n_1, n_2)$.

From the above properties, we have

$$f_1(n) + f_2(n) \leq 2(\max(|f_1(n)|, |f_2(n)|))$$
\[ \leq 2(c_* \max(|g_1(n)|, |g_2(n)|)) \]
\[ \leq 2c_* (\max(|g_1(n)|, |g_2(n)|)) \]

for all \( n \geq n_* \) for \( 2c_* \geq 0 \). Therefore, we have shown that \( f_1(n) + f_2(n) \in O(\max(|g_1(n)|, |g_2(n)|)) \).

**Problem 5**

Prove by showing both sides count the same thing that

\[ (n + 1)(n!) = \sum_{k=0}^{n} \binom{n}{k} k! (n-k)! \]

for \( n \geq 1 \).

**Hint:** Consider a set of size \( n \) and some fixed \( k \) between 0 and \( n \). What does \( \binom{n}{k} \) indicate in this situation? What does \( k!(n-k)! \) represent?

**Solution**

We will show that both sides count the number of ways to partition \( n \) elements where the order matters within each partition

a. left side: If we take the \( n \) elements and order them, there are \( n! \) total ways of ordering. Then when we partition the set, we can place the divider in and \( n + 1 \) slots. so the total ways of partitioning \( n \) elements where the order matters is \( n!(n+1) \)

b. right side: The other way to count this is to sum up all the different possibilities for each size of the partition. So if all \( n \) elements are in the first partition, there are \( n! \) ways of ordering; in general, if \( k \) of \( n \) elements are in the first partition, there are \( \binom{n}{k} \) ways of splitting the partition; \( k! \) ways of ordering the first partition and \( (n-k)! \) ways of ordering the second partition. So overall for each possible \( k \), there is \( \binom{n}{k} k!(n-k)! \) of ordering it. and because \( k \) can be anything from 0 elements in the first partition to all \( n \) elements in the first partition we sum from \( k = 0 \) to \( k = n \). So the total is \( \sum_{k=0}^{n} \binom{n}{k} k!(n-k)! \)

So we have shown both sides count the same value, so they are equal.