Problem 1

Consider a circuit with $n \geq 0$ input wires and one output wire. Let the wires of the circuit be represented by the set $I_n = \{w_1, w_2, \ldots, w_n\}$, where $w_i$ represents the $i$th wire. Consider some subset $t \subseteq I_n$. We say that $t$ satisfies the circuit if the circuit evaluates to true when all input wires in $t$ are set to true and all input wires not in $t$ are set to false.

Let $T_n$ be the set of all $t \subseteq I_n$ such that $t$ satisfies the circuit. Prove that for any $n \geq 2$, there exists a circuit where:

a. $|T_n| = 1$

b. $|T_n| = 2^n - 1$

c. $|T_n| = 2^n$

d. $|T_n| = 2^{n-1}$

e. $|T_n| = n$

Note: You do not need to use induction or diagram your circuits for this problem. You must use every wire in each circuit. Also do not have more than two inputs into any gates.

Problem 2

An $n$-bit boolean function $f$ maps a 0/1 string of length $n$ to either 0 or 1, for $n > 0$.

a. Count the number of $n$-bit boolean functions.

b. How many $n$-bit boolean functions are bijective?
An \( n \)-bit boolean function \( \text{depends} \) on the position \( i \) in the string if there exists two strings \( A \) and \( B \) such that \( A \) and \( B \) differ only in position \( i \) and \( f(A) \neq f(B) \).

c. Count the number of \( n \)-bit boolean functions that do not depend on bit \( i \).

d. For \( n > 1 \), count the number of \( n \)-bit boolean functions that map a string to 1 if (not iff) its first bit, second bit, or both are 1.

Let \( A = (a_1, a_2, \cdots, a_n) \in \{0,1\}^n \) be a 0/1 string of length \( n \) where \( a_i \) represents the bit at position \( i \). A boolean function \( f \) is \textit{symmetric} if for all \( A \) in \( \{0,1\}^n \), \( f \) maps \( A \) and all of the strings corresponding to each of the possible permutations of \( A \) to the same element in the codomain.

e. Count the number of \( n \)-bit symmetric boolean functions.

**Problem 3**

On a \( 4 \times 4 \) square grid of 16 dots, how many paths of length six connect the lower left-hand corner dot to the upper right-hand corner dot (dot \( a \) to dot \( b \))? The length of a path is the number of hops between points on that path. A hop can be a move up, move down, move left, or move right.

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & b \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
a & \bullet & \bullet & \bullet \\
\end{array}
\]

**Problem 4**

Prove that \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \) and use this result to prove that \( \sum_{k \in \text{odd}} \binom{n}{k} = \sum_{k \in \text{even}} \binom{n}{k} \).

\textbf{Note:} You have actually already proved this using a bijection on previous homework. This is just another way to prove something you already know!
Problem 5

Show that the LHS and RHS in the following equation count the same thing. Do not use any algebraic manipulation in your argument.

\[
\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}
\]