Note that the due date for this assignment is the day before your second midterm. This deadline is an extension (giving everybody until what would typically be the late deadline for this assignment) to be as accommodating as possible with spring break and the upcoming midterm, but be aware of the midterm date and do not leave this assignment to the last minute!

This assignment is due at 2:30 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

Problem 1

Jim notices that Dwight has left his computer unattended and decides that it would be funny to change Dwight’s background to a picture of Jell-O. Unfortunately, the computer is password protected. Determined to carry out his prank, Jim attempts to guess Dwight’s password.

a. Dwight’s password is 8-10 characters long, and contains only alphanumeric characters (any of the 26 upper-case letters, 26 lower-case letters, and 10 digits). Count the number of distinct passwords.

b. Dwight’s password is 8 characters long but can also contain the following special characters: ! @ # $ % + * ( ). Jim remembers that Dwight’s favorite number is two, which leads him to believe Dwight’s password will contain exactly 2 characters from each of the above types (upper-case letters, lower-case letters, digits, and special characters). Under this assumption, how many distinct passwords are possible?

Problem 2

Dunder Mifflin Paper Company is having a company picnic where employees from all of the branches come and compete in fun games. At the picnic, the employees are
given a hat, t-shirt, and bandana. There are 10 different colors each of hats, t-shirts, and bandanas.

a. David Wallace, the head of Dunder Mifflin, decides that he does not like seeing blue and red (both of which are included in the 10 colors of hats, t-shirts and bandanas) in the same outfit.

How many outfits satisfy David Wallace's taste and are also not all the same color?

b. Michael only wants outfits that are made of exactly three different colors and also do not have both red and blue.

How many outfits satisfy Michael?

c. Ryan wants to make sure he looks really cool at the company picnic, so on his way there he stops at the jewelry store. They have fifteen different ring colors available. Naturally, he wants to buy one ring for each finger (for a purchase of ten rings total). He wants to have at least three colors of rings.

How many ways can he choose rings such that no two adjacent fingers have the same color? **Note:** His thumbs are not considered adjacent to each other.

**Problem 3**

Ryan realizes that in order to become the youngest CEO of Dunder Mifflin ever, he needs to prove he is a math whiz. However, he isn’t. Help him complete the following quiz posted by corporate.

a. Define a set of positive integers to be *selfish* if it contains its own cardinality. A set is *minimally selfish* if it is selfish, but it has no proper subsets which are also selfish. Count the number of minimally selfish subsets of \{1, 2, \ldots, n\}.

b. Suppose the prime factorization of some \(n \in \mathbb{Z}^+\) is \(n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}\). Count the number of distinct positive integers that divide \(n\).

**Problem 4**

a. Using the binomial theorem, prove that \(\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\).

b. Use the result from Part A to conclude that the number of odd-cardinality subsets is equal to the number of even-cardinality subsets (for any arbitrary \(n\) element set).
Problem 5

An $n$-bit boolean function $f$ maps a 0/1 string of length $n$ to either 0 or 1.

a. Count the number of $n$-bit boolean functions. Write your expression in terms of $n$.

b. How many $n$-bit boolean functions are bijective?

An $n$-bit boolean function depends on the position $i$ in the string if there exists two strings $A$ and $B$ such that $A$ and $B$ differ only in position $i$ and $f(A) \neq f(B)$.

c. Count the number of $n$-bit boolean functions that do not depend on bit $i$.

d. For $n > 1$, count the number of $n$-bit boolean functions that map a string to 1 if (not iff) its first bit, second bit, or both are 1.

Let $A = (a_1, a_2, \cdots, a_n) \in \{0, 1\}^n$ be a 0/1 string of length $n$ where $a_i$ represents the bit at position $i$. A boolean function $f$ is symmetric if for all $A$ in $\{0, 1\}^n$, $f$ maps $A$ and all of the strings corresponding to each of the possible permutations of $a_1, a_2, \cdots, a_n$ to the same element in the codomain.

e. Count the number of $n$-bit symmetric boolean functions.

Problem 6

a. Michael wants to guard his newest film script from Dwigt, so he orders a safe room with a pressure-sensitive tile floor that is represented by integer Cartesian points. The safe room will set off an alarm if the tiles are not stepped on in the correct order. Michael wants to create a single safe path through the room.

His floor technician says only some paths can be safe. Paths cannot go diagonally between tiles, nor can they move more than one tile in a single step. Further, to simplify the electronics, Michael must specifically select a path from $(0, 0)$ to $(n, n)$ such that it only moves up and right and does not contain a point $(x, y)$ such that $x < y$. Count the number of possible paths that Michael can make.

b. A competing safe room installation company advertises that they offer safes that are exactly the same as the one Michael installed, except that all paths formed on the tiles must be paths from $(0, 0)$ to $(n + 1, n - 1)$ that only move up and right, and do not contain a point $(x, y)$ such that $x \leq y$ (except for $(0, 0)$). Prove that this new safe room has the same number of valid paths as Michael’s by showing the existence of a bijection between them.

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\(^1\)Points in the integer plane, i.e. $\{(x, y) \mid x, y \in \mathbb{Z}\}$.
(Hint: Drawing a diagram to familiarize yourself with valid paths can be helpful. Diagrams are an acceptable supplement to a proof, but never a replacement for one.)