All homeworks are due at 12:55 PM on Gradescope.
Please do not include any identifying information about yourself in the handin, including your Banner ID.
Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

For each of the following expressions, convert it to CNF. Using truth tables or logical identities, prove your converted expression is logically equivalent to the given one.

Conjunctive Normal Form (CNF) is a particular way of writing logical expressions. We define it as follows:

- A literal is a variable or its negation. \( x_1 \) and \( \neg x_1 \) are examples of literals
- A clause is any disjunction (OR) of any number of literals (for example, \( (x_1 \lor x_2) \)). A literal by itself is also a clause.
- An expression is in conjunctive normal form if it is the conjunction (AND) of any number of clauses (for example, \( (x_1 \lor x_2) \land (x_3 \lor \neg x_1) \)).

a. Expression: \( (p \land \neg q) \lor \neg p \)
b. Expression: \( (p \land \neg q) \lor (\neg p \land q) \)
c. Expression: \( \neg(p \land (q \lor r)) \)
d. Expression: \( (p \implies q) \implies r \)

Problem 2

For each of the following, answer the questions and provide your reasoning.

a. Can AND (\( \land \)) be expressed using only OR (\( \lor \)) and NOT (\( \neg \))?
b. A set of Boolean operations is *toasty* if all possible truth tables can be expressed using only the operators in that set. Is the set \{OR, AND\} toasty?
c. Is the set \{OR, AND, NOT\} toasty?

\textbf{Hint:} Consider the truth table representing an arbitrary proposition. Is there a way that you could express this table in terms of the above operators?

d. Consider the ♠ operator, defined by \( p \diamond q = \neg(p \lor q) \). Is the set \{♠\} toasty?

\textbf{Problem 3}

a. Suppose \( p \) and \( q \) are propositions such that \( p \Rightarrow q \) is False. Determine the truth values of:

(i) \( \neg p \Rightarrow q \)
(ii) \( p \lor q \)
(iii) \( q \Rightarrow p \)

b. You are now going to design a circuit that takes as input two 1-bit binary numbers \( A \) and \( B \) and outputs whether or not \( A > B \), \( A < B \), or \( A = B \). Namely, the circuit should have two inputs, the bits \( A \) and \( B \), and three outputs \( G \), \( E \), and \( L \). For any input, exactly one of \( G \) (greater), \( E \) (equal), or \( L \) (less) should be 1. Note that \( G \) corresponds to \( A > B \).

(i) Write out a truth table equivalent to this circuit.
(ii) Draw your circuit. Please use only AND, OR, and NOT gates with at most two inputs per gate. Note that we care only about the correctness of the circuit, not its complexity.

Be sure to explain how your circuit works.

\textbf{Problem 4}

a. Although we have been using AND and OR gates with only two inputs, we can create AND and OR gates that take in more than two inputs. Under the hood, these can just be implemented with 2-input AND and OR gates. You are now going to build a 3-input AND circuit using only 2-input AND gates, and build a 3-input OR circuit using only 2-input OR gates.

b. Reuben has developed a secret sandwich recipe that must be protected. Help secure Reuben’s filing cabinet with an electronic lock!

The password will be represented as a three bit integer, using three input wires. When the correct password is entered into the wires, the output wire will activate to unlock the cabinet.
Draw a circuit with three “password entry” input wires for inputting the password, and three more “password definition” wires whose state indicates what the password actually is. When these match pairwise, the password wire should activate.

For this problem you may use all logic gates and gates with more than two inputs. Be sure to explain how your circuit works.

Problem 5

a. The Sand Witch has spent many nights trying to determine which foods are sandwiches and which are not. They decide to create a poll to determine what people think. However, to do so, they need a machine that can handle the poll data.

To help them out, construct a circuit using only AND, OR, and NOT gates. You may use gates of any number of inputs for this problem.

The circuit will take in three people’s answers (1 for yes and 0 for no), and return the answer of the majority. Be sure to explain how your circuit works.

b. The Sand Witch is currently having a lot of trouble determining if the numbers 0, 1, 2, and 3 are odd or even. The solution? Build a circuit!

Model this problem as a circuit using only AND, OR, and NOT gates. You may use gates with more than one input. Create three input wires and one output wire. If an even number of inputs are on, the output should be off. If an odd number of input wires are on, the output should be on. Be sure to explain how your circuit works.

Note: The output depends only on how many of the input wires are on. For example, if exactly one input is on, then the output should be on. It does not matter which input is on.