All homeworks are due at 12:55 PM in the CS22 bin on the CIT second floor, next to the Fishbowl.

Include our cover sheet or equivalent, write your Banner ID (but not your name or your CS login) on each page of your homework, label all work with the problem number, and staple the entire handin before submitting.

Be sure to fully explain your reasoning and show all work for full credit. Consult the style guide for more information.

**Problem 1**

a. Let $A$ be a set with $n$ elements. Let $T$ be the set of ordered pairs $(X,Y)$ where $X$ and $Y$ are subsets of $A$. Let $S$ be the set of 0/1/2/3 strings of length $n$. That is, elements of $S$ are strings of length $n$ where each character is 0, 1, 2, or 3. Give (and prove) a bijection between $T$ and $S$.

b. If there exists a bijection between two finite sets $M$ and $N$, what can you conclude about the sizes of $M$ and $N$?

**Problem 2**

a. Prove by induction that for all positive integers $n$, there exists a positive integer $m$ such that:

$$m^2 \leq n < (m + 1)^2$$

b. Prove by contradiction that there exists a **unique** such $m$.

**Problem 3**

a. Prove by contradiction that for any integer $n$, $n^2 - 2$ is not divisible by 4.

b. Prove that for any integer $n$, $n^3$ is odd if and only if $n$ is odd.

c. Prove that for any integer $n$, $n^3 - n$ is divisible by 3.
Problem 4

Consider the following relation on the set of integers:
\[ \forall a, b \in \mathbb{Z}, \ (a, b) \in R \text{ if and only if the remainder when } a \text{ is divided by 3 is the same as the remainder when } b \text{ is divided by 3.} \]

a. Prove that \( R \) is an equivalence relation.

b. How many distinct equivalence classes are in this equivalence relation? What are they?

Note: An equivalence class is defined for an equivalence relation, \( R \) on set \( A \), as follows: \([a]_R = \{x \in A \mid (a, x) \in R\}\)

Problem 5

This is an optional problem. It will not affect your grade.

Let \( C(n) \) be the number of 0/1 strings of length \( n \) that do not contain consecutive 1s. For example, \( C(4) = 8 \) because there are 8 0/1 strings of length 4 without consecutive 1s: 0000, 0001, 0010, 0100, 1000, 0101, 1010, and 1001.

Prove that \( \forall n \in \mathbb{Z}^+, \ C(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \right) \).

Hint: Show that for any \( n \geq 3 \), \( C(n) = C(n-1) + C(n-2) \). Also, you may use the following without proof:

- \( 1 + \frac{1+\sqrt{5}}{2} = \left( \frac{1+\sqrt{5}}{2} \right)^2 \)
- \( 1 + \frac{1-\sqrt{5}}{2} = \left( \frac{1-\sqrt{5}}{2} \right)^2 \)