Problem 1

Assume that all sets mentioned in this problem are subsets of a universal set $U$. Prove the following:

a. For all sets $A$ and $B$, $A \cap B = (A \cup B) \setminus (A^c \cup B^c)$
b. For all sets $A$, $B$, and $C$, $(A \setminus C) \cap (B \setminus C) \cap (A \setminus B) = \emptyset$.

Problem 2

A relation $R$ on a set $S$ is **antisymmetric** if for any elements $x, y \in S$, $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$. In other words, if $x$ is related to $y$, then $y$ is not allowed to be related to $x$ unless $x$ is $y$.

a. Find a non-empty set $S$ and a relation $R$ on $S$ such that $R$ is neither symmetric nor antisymmetric.
   i. Give $S$ and $R$.
   ii. Justify why your relation is neither symmetric nor antisymmetric.
   iii. What is the smallest possible cardinality of $S$ over which $R$ is neither symmetric nor antisymmetric? Justify your answer.

b. If a relation, $R$, is both symmetric and antisymmetric, is $R$ necessarily reflexive? Justify your answer.

Problem 3

a. Prove that a relation that is reflexive and symmetric need not also be transitive.
b. Prove that a relation that is reflexive and transitive need not also be symmetric.
c. Prove that a relation that is symmetric and transitive need not also be reflexive.
Problem 4

a. i. Find all relations on \{0, 1\} × \{1\}.
   ii. Find all relations on \{1\} × \{0, 1\}.

b. i. Find all functions from \{0, 1\} to \{1\}.
   ii. Find all functions from \{1\} to \{0, 1\}.

c. Let \(S = \{0, 1\}\), \(T = \{t | t \subseteq S \times S\}\), and \(R\) be the set of all possible functions from \(S\) to \(S\).
   i. Can an injection from \(T\) to \(R\) exist? If so, give one such injection and prove that this mapping is indeed injective. If not, prove why such a mapping cannot exist.
   ii. Can a surjection from \(T\) to \(R\) exist? If so, give one such surjection and prove that this mapping is indeed surjective. If not, prove why such a mapping cannot exist.
   iii. Can a bijection from \(T\) to \(R\) exist? If so, why? If not, why not?

Problem 5

Let \(A\) be a set with \(n\) elements. Let \(T\) be the set of all ordered pairs \((X, Y)\) where \(X\) and \(Y\) are subsets of \(A\). Let \(S\) be the set of 0/1/2/3 strings of length \(n\). That is, elements of \(S\) are strings of length \(n\) where each character is 0, 1, 2, or 3. Give (and prove) a bijection between \(T\) and \(S\).

Conclude that \(T\) and \(S\) must be the same size.