Expanding Stacks & Queues

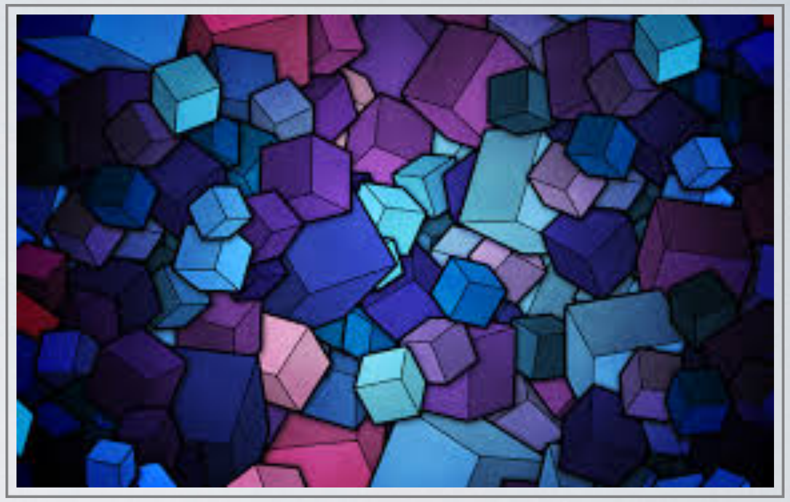
CS16: Introduction to Data Structures & Algorithms

Summer 2021

Outline

- Abstract data types
- Stacks
  - Capped-capacity
  - Expandable
- Amortized analysis
- Queues
  - Expandable queues

Abstract Data Types



- Abstraction of a data structure
- Specifies "functionality"
  - type of data stored
  - operations it can perform
- Like a Java interface
  - Specifies name & purpose of methods
  - But not implementations
- Think of lists: can have ArrayLists or LinkedLists

Stacks

- Stores arbitrary objects
- Operations
  - **Push**: adds object
  - **Pop**: returns *last* object
  - LIFO: last-in first-out
- Can be implemented with
  - Linked lists, arrays, …

Stack ADT

- **`push`**`(object)`
  - inserts object
- `object` **`pop`**`( )`
  - returns and removes last inserted object
- `int` **`size`**`( )`
  - returns number objects in stack
- `boolean` **`isEmpty`**`( )`
  - returns **`TRUE`** if empty; **`FALSE`** otherwise

Capped-Capacity Stack

- Array-based Stack
  - Stores objects in array
  - keeps pointer to last inserted object
- Problem?
  - Size of the stack is bounded by size of array :-(

Capped-Capacity Stack

```
Stack( ):
  data = array of size 20
  count = 0
```

O(1)

```
function size( ):
  return count
```

O(1)

```
function isEmpty( ):
  return count == 0
```

```
function push(object):
  if count < 20:
    data[count] = object
    count++
  else:
    error("overfull")
```

O(1)

```
function pop( ):
  if count == 0:
    error("empty stack")
  else:
    count--
    return data[count]
```

O(1)

Expandable Stack

- Capped-capacity stack is fast
  - but—what if we don't know how many items?
- How can we design an *uncapped* Stack?
- Remember—arrays can't be resized
  - "resize"—copy contents of size N array to size N' array

Expandable Stack

- Strategy #1: **Incremental**
  - increase size of array by constant **e** when full
- Strategy #2: **Doubling**
  - double size of array when full

Expandable Stack

```
Stack( ):
  data = array of size 20
  count = 0
  capacity = 20
```

What is the runtime?

```
function push(object):
  data[count] = object
  count++
  if count == capacity
    new_capacity = capacity + e /* incremental */
                 = capacity * 2 /* doubling */
    new_data = array of size new_capacity
    for i = 0 to capacity - 1
      new_data[i] = data[i]
    capacity = new_capacity
    data = new_data
```

Expandable Stack

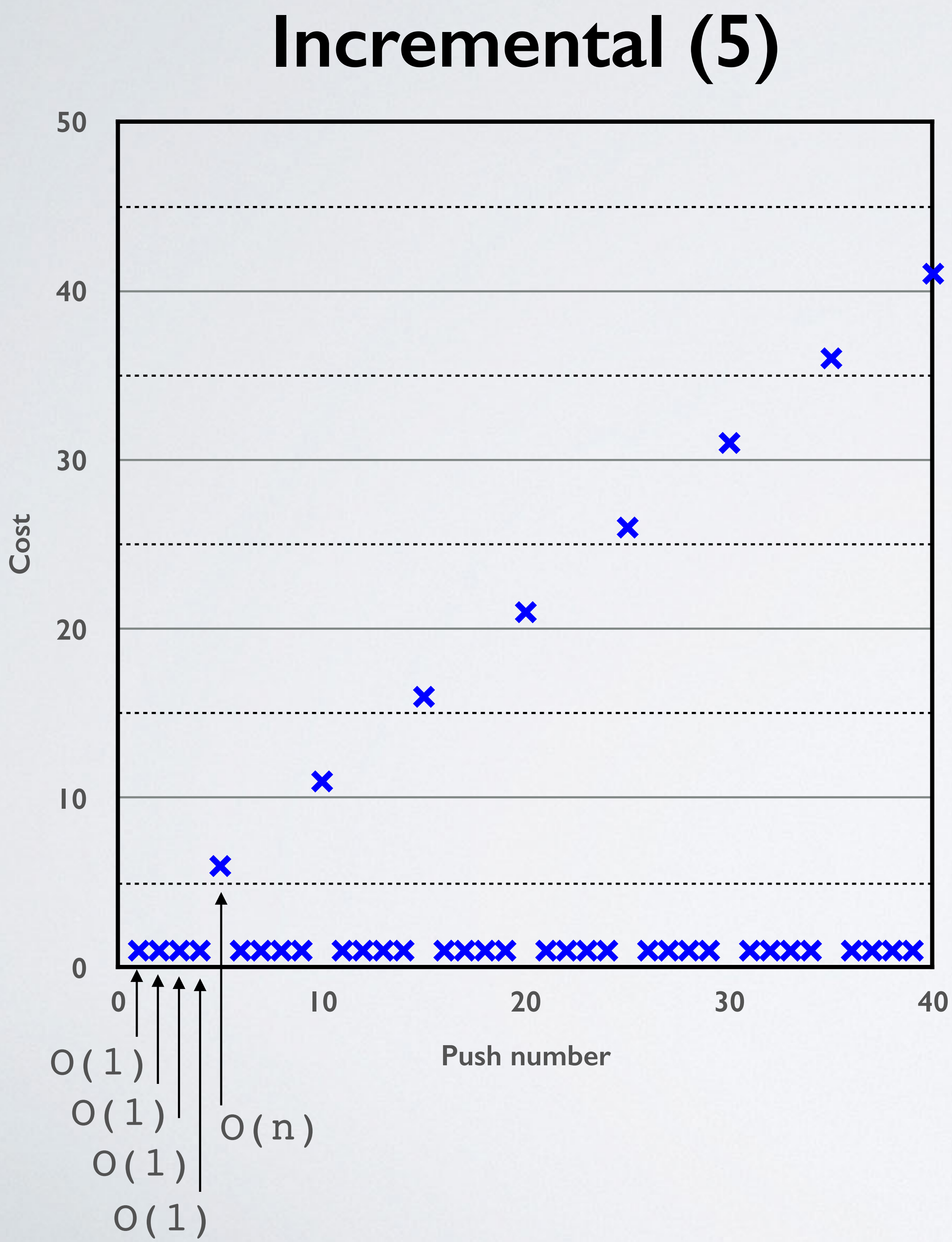
```
function push(object):
  data[count] = object
  count++
  if count == capacity
    new_capacity = capacity + e /* incremental */
                 = capacity * 2 /* doubling */
    new_data = array of size new_capacity
    for i = 0 to capacity - 1
      new_data[i] = data[i]
    capacity = new_capacity
    data = new_data
```

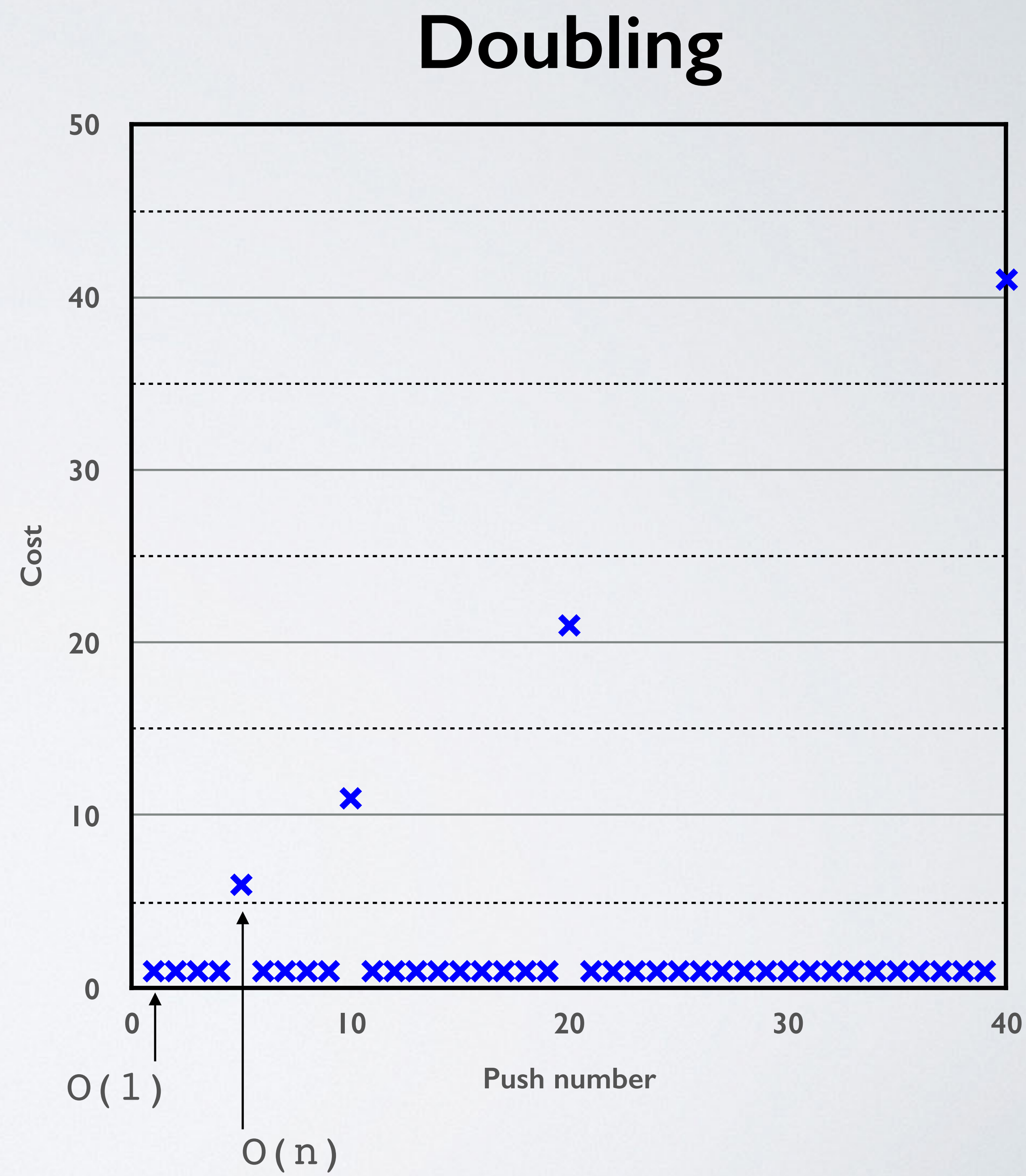
- Runtime when not expanding is `O(1)` & runtime when expanding is `O(n)`
- When does it expand?
  - after **n** pushes, where **n** is capacity of array

Incremental & Doubling

- What are the *worst-case* runtimes?
  - incremental: `O(n)`
  - doubling: `O(n)`
- But are they really the same?

Incremental & Doubling





Incremental & Doubling

- Worst-case analysis overestimates runtime
  - for algorithms that are fast *most* of time…
  - …and slow *some* of the time
- For these algorithms we need an alternative
  - Amortized analysis!

Measure cost on <u>*sequence*</u> of calls not a single call!

Towards Amortized Analysis

- For certain algorithms it's better to measure
  - *total* running time on *sequence* of calls
  - instead of measuring on a single call
  - `S(n)`: total #calls on *sequence* of `n` calls
  - **Not runtime on a single input of size n**
- For a stack
  - `S(n)`: cost push `#1` + cost push `#2` + … + cost push `#n`

Amortized Analysis

- Instead of reporting *total* cost of sequence
  - report cost of sequence *per call*

$$\boxed{\frac{S(n)}{n}}$$

Average Joe's GYM

Amortized Analysis of Incremental

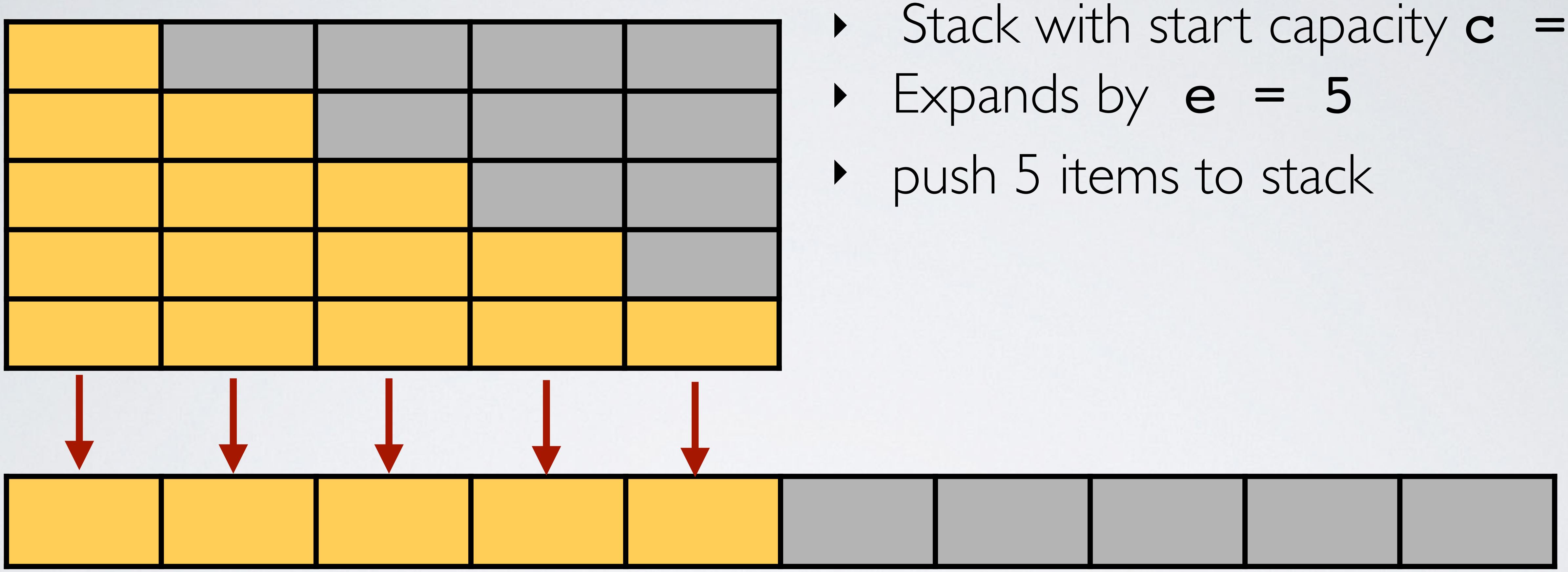
Expandable Stack

```
Stack( ):
  data = array of size 20
  count = 0
  capacity = 20
```

Run time depends on count which depends on # of *previous* pushes

```
function push(object):
  data[count] = object
  count++
  if count == capacity
    new_capacity = capacity + e /* incremental */
                 = capacity * 2 /* doubling */
    new_data = array of size new_capacity
    for i = 0 to capacity - 1
      new_data[i] = data[i]
    capacity = new_capacity
    data = new_data
```

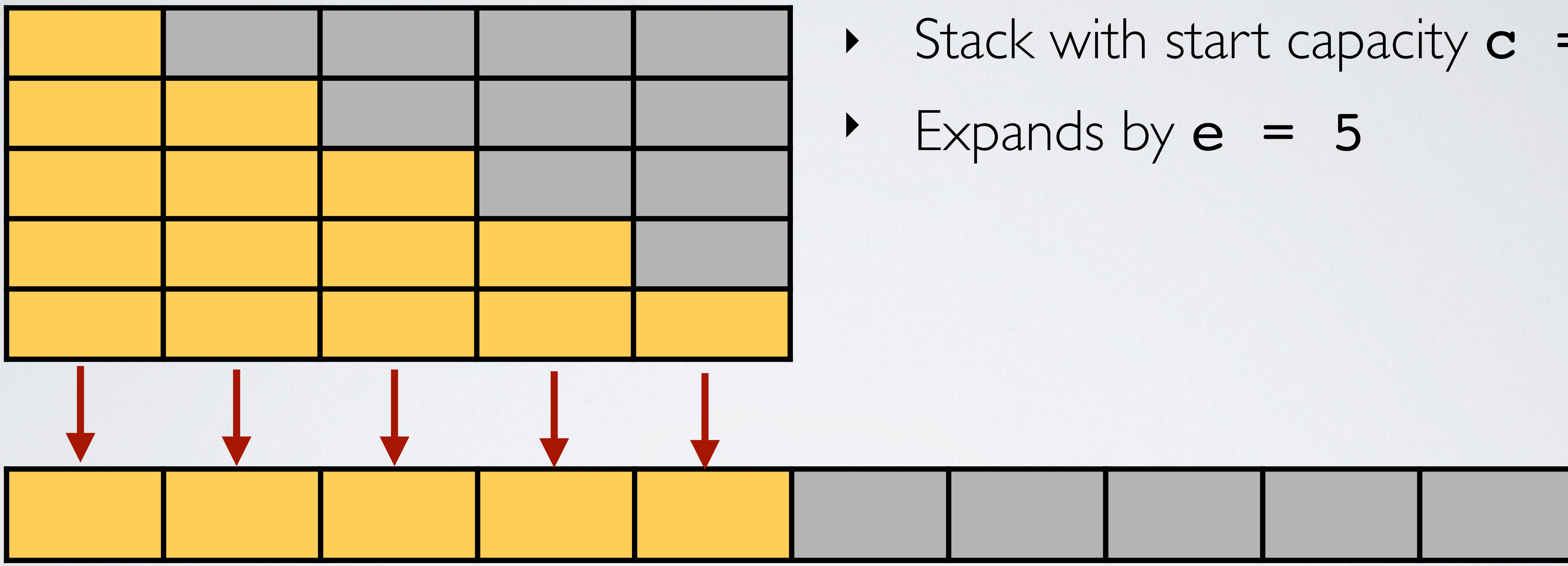
Amortized Analysis of Incremental



- Stack with start capacity `c = 5`
- Expands by `e = 5`
- push 5 items to stack

- 5th push brings to capacity
  - Objects copied to new array of size `c+e = 5+5 = 10`
  - Cost per push over **5** pushes?

Amortized Analysis of Incremental



- Stack with start capacity `c = 5`
- Expands by `e = 5`

$$\frac{S(n)}{n} = \frac{c + c}{c} = \frac{5+5}{5} = 2$$

pushes

expansion

Is each push `O(1)`?

Amortized Analysis of Incremental

- What if we push `10` objects?

$$\frac{S(n)}{n} = \frac{\boxed{c} + \boxed{c} + \boxed{e} + \boxed{(c+e)}}{10}$$

1st batch of pushes

1st expansion

2nd batch of pushes

2nd expansion

pushes

expansions

c=5
e=5

Amortized Analysis of Incremental

- What if we push `10` objects?

$$\frac{S(n)}{n} = \frac{c + c + e + (c+e)}{10}$$

$$= \frac{c + e + c + (c+e)}{10}$$

$$= \frac{10 + 5 + (5+5)}{10}$$

$$= 2.5$$

pushes

expansions

c=5
e=5

Amortized Analysis of Incremental

$$\frac{S(10)}{10} = \frac{c+e+c+(c+e)}{10} = \frac{10+5+10}{10} = \frac{25}{10} = 2.5$$

$$\frac{S(15)}{15} = \frac{c+e+e+c+(c+e)+(c+e+e)}{15} = \frac{15+5+10+15}{15} = \frac{45}{15} = 3$$

$$\frac{S(20)}{20} = ?$$

pushes

expansions

c=5
e=5

Amortized Analysis of Incremental

$$S(n) = c + e + \ldots + e + c + (c + e) + (c + 2e) + (c + 3e) + \ldots$$

$$= n + c + (c + e) + (c + 2e) + (c + 3e) + \ldots$$

To make things simpler, let's assume
`e = c`

$$= n + c + 2c + 3c + 4c + \ldots + \frac{n}{c} \cdot c$$

# of expansions
(`1` expansion per `c` (or `e`) pushes)

Amortized Analysis of Incremental

n pushes

cost of exp. # n/c

$$S(n) = n + c + 2c + 3c + \cdots + \frac{n}{c} \cdot c$$

$$= n + c \cdot \left(1 + 2 + \cdots + \frac{n}{c}\right)$$

factoring out c

$$= n + c \cdot \frac{1}{2} \cdot \left(\frac{n}{c}\left(\frac{n}{c} + 1\right)\right)$$

using:

$$\boxed{(1 + 2 + \cdots + k) = \frac{k \cdot (k+1)}{2}}$$

$$= n + \frac{n^2/c + n}{2}$$

distributing & simplifying:

$$= O(n^2)$$

$$\boxed{\frac{S(n)}{n} = O(n)}$$

Amortized Analysis of Incremental

- Summary
  - Total cost of **n** pushes: `S(n) = O(n²)`
  - Amortized cost of **n** pushes: `S(n)/n = O(n)`

Amortized Analysis of Double

Amortized Analysis of Doubling

- Doubling stack with initial capacity **c=5**?

cost of pushes

$$\frac{S(n)}{n} = \frac{S(5)}{5} = \frac{5+5}{5} = 2$$ **cost of exp**

$$\frac{S(n)}{n} = \frac{S(10)}{10} = \frac{10+5+10}{10} = 2.5$$ **cost of exp #2**

$$\frac{S(n)}{n} = \frac{S(20)}{20} = \frac{20+5+10+20}{20} = 2.75$$ **cost of exp #3**

Amortized Analysis of Doubling

cost of n pushes

cost of last exp

cost of second to last exp

cost of exp #1

$$\begin{aligned} S(n) &= n + n + \frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^{k-1}} \\ &= n + n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) \\ &< n + n \cdot 2 \\ &= 3n \end{aligned}$$

using: $\boxed{\lim_{k \to \infty} \sum_{i=0}^{k} \frac{1}{2^i} = 2}$

Assume:
`c=2`
`n=2`^{k}

$$\boxed{\frac{S(n)}{n} = O(1)}$$

Amortized Analysis

- Summary for Incremental
  - Total cost of **n** pushes: $S(n) = O(n^2)$
  - Amortized cost of **n** pushes: $S(n)/n = O(n)$
- Summary for Doubling
  - Total cost of **n** pushes: $S(n) = O(n)$
  - Amortized cost of n pushes: $S(n)/n = O(1)$

Amortized Analysis

- Summary for Incremental
  - Total cost of **n** pushes: $S(n) = O(n^2)$
  - Amortized cost of **n** pushes: $S(n)/n = O(n)$
- Summary for Doubling
  - Total cost of **n** pushes: $S(n) = O(n)$
  - Amortized cost of n pushes: $S(n)/n = O(1)$
- In practice: always use doubling

How do we feel about amortized analysis?

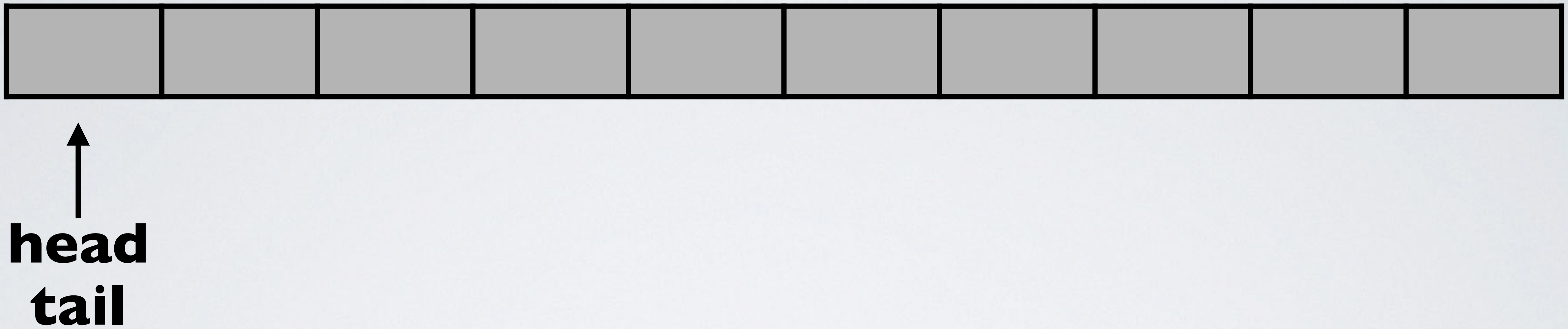
- Situations where worst case is most important?

Expandable Queue

Queue ADT

- **`enqueue`**`(object)`:
  - inserts object
- `object` **`dequeue( )`**
  - returns and removes first inserted object
- `int` **`size( )`**
  - returns number objects in queue
- `boolean` **`isEmpty( )`**
  - returns **`TRUE`** if empty; **`FALSE`** otherwise

Expandable Queue



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  - need to keep track of head and tail

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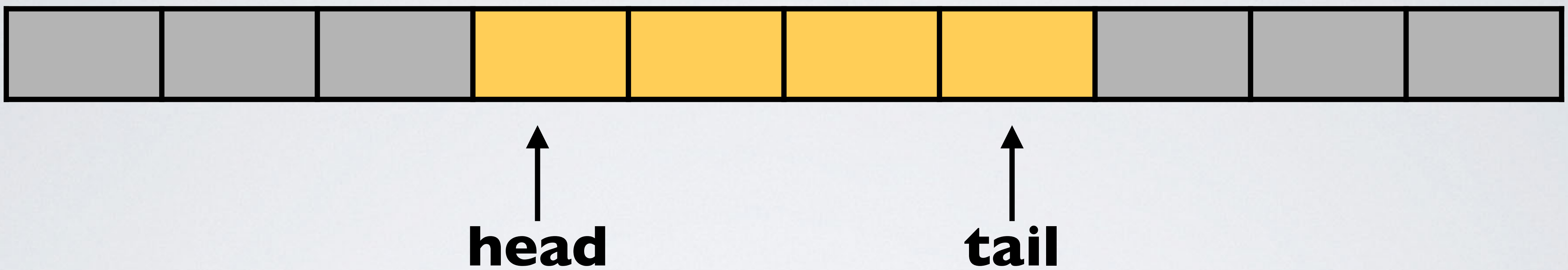
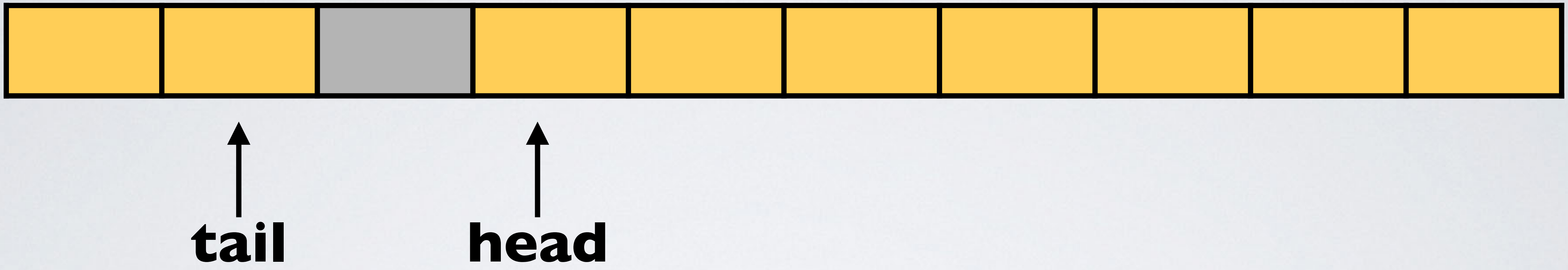


- Can be implemented with expandable array
  - need to keep track of head and tail
- What happens when tail reaches end?
  - Is the queue full?
- So when should we expand array?

Expandable Queue



- Wrap around until array is completely full
- When expanding re-order objects properly

Expandable Queue

```
function enqueue(object):
  if size == capacity
    double array and copy contents
    reset head and tail pointers
  data[tail] = object
  tail = (tail + 1) % capacity
  size++
```

$$\frac{S(n)}{n} = O(1)$$

```
function dequeue( ):
  if size == 0
    error("queue empty")
  element = data[head]
  head = (head + 1) % capacity
  size--
  return element
```