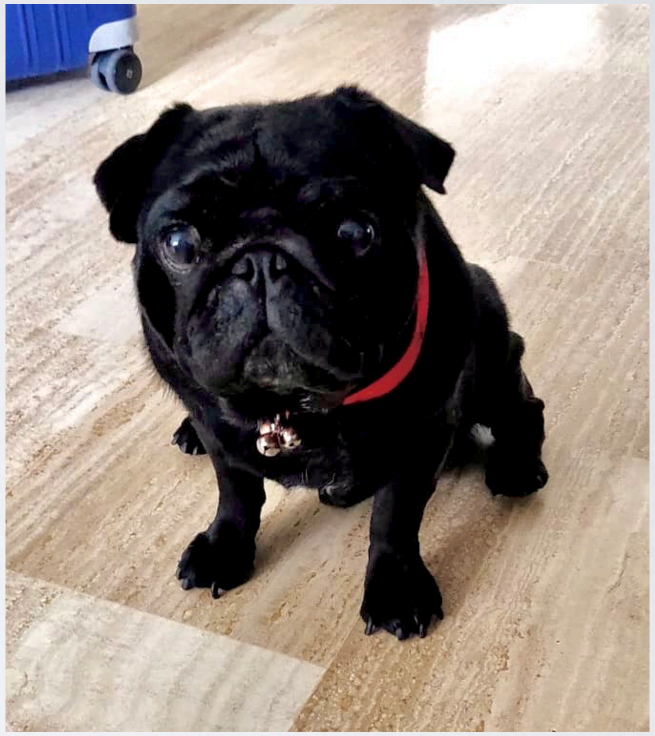
Online Algorithms and Competitive Analysis

CS16: Introduction to Data Structures & Algorithms

Spring 2020

Outline

1. Motivation
2. The Ski-Rental Problem
3. Experts Problem
4. Dating Problem



Motivation

- We don't always start off with all the data we want at once
- We want the best algorithms to answer questions about such data

Offline Algorithms

- An ***offline algorithm*** has access to all of its data at the start – it "knows" all of its data in advance
- Most of what you have done in this class has been offline (or at least given offline)

Online Algorithms

- An ***online algorithm*** does not have access to all of the data at the start
- Data is received **serially**, with no knowledge of what comes next
- How do you make a good algorithm when you don’t know the future?

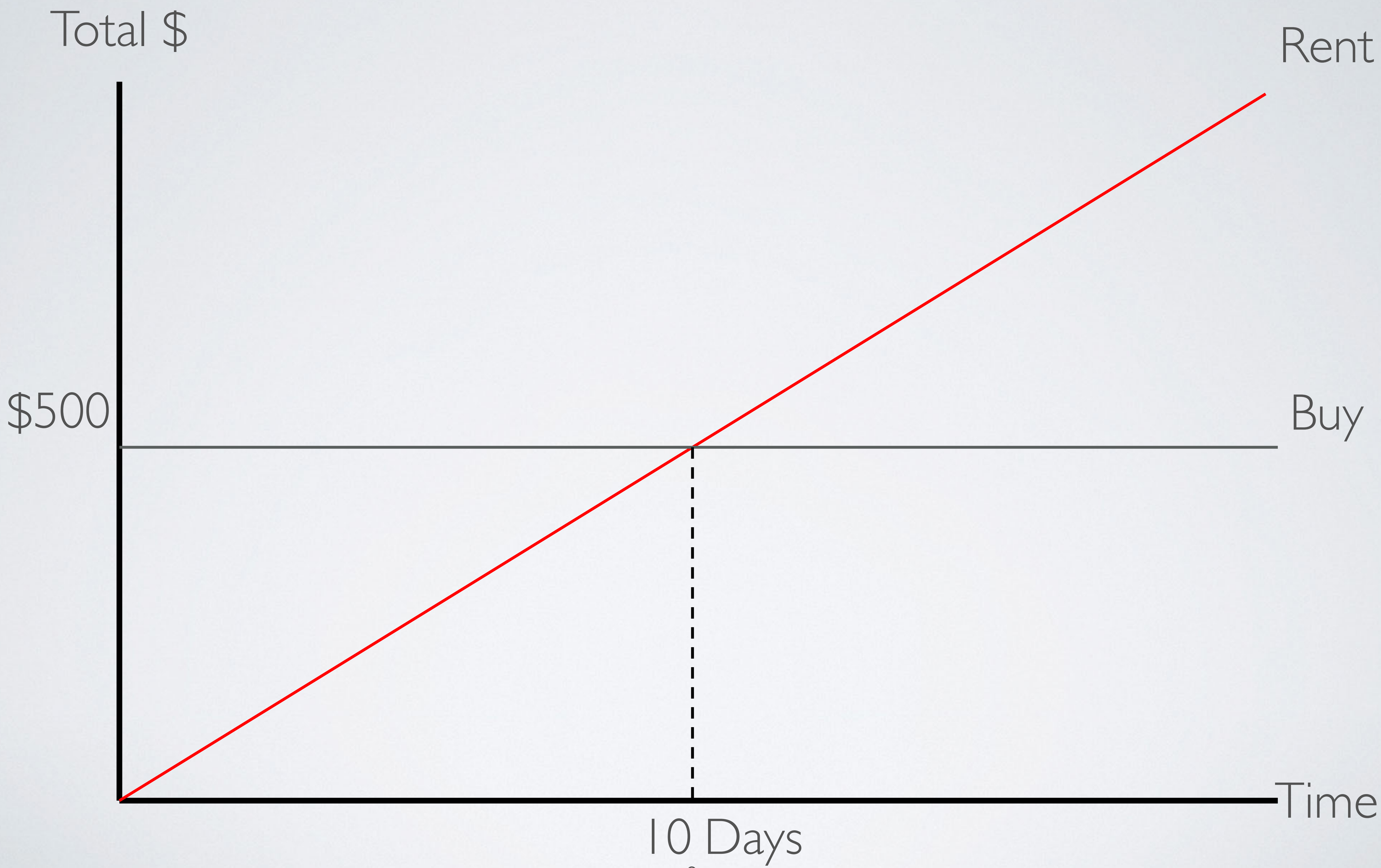
Ski-Rental Problem

- You like skiing
- You're going to go skiing for **n** days
- You need to decide: Do you rent skis or buy skis?
- Renting:
  - $50 per day
- Buy:
  - $500 once
- Goal: Minimize cost

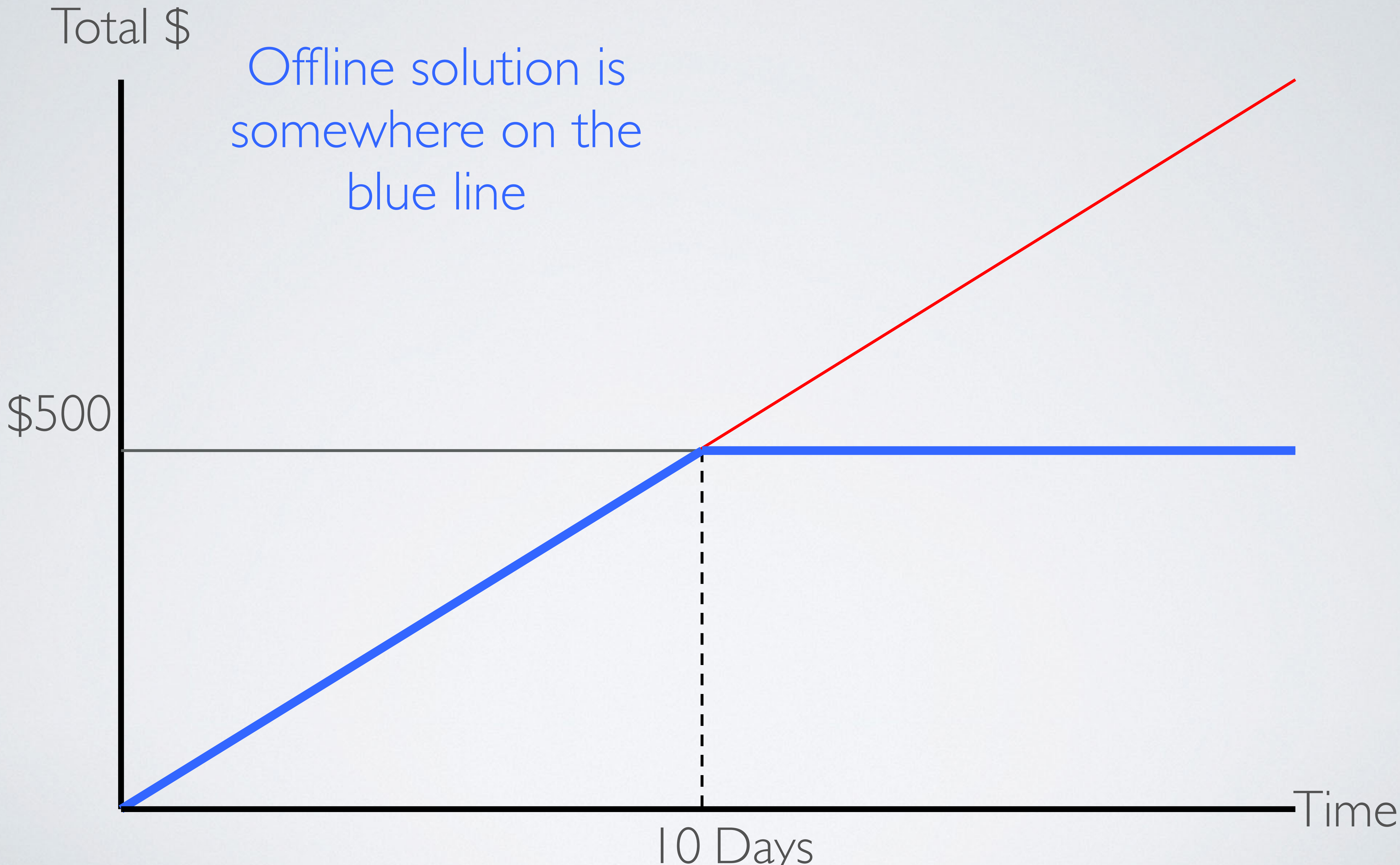
Ski-Rental Problem (Offline)

- Offline solution:
  - If $n < 10$, rent!
  - Else, buy!
- Tough luck. You don't know what n is
  - You love skiing so much, you'll ski as long as you don't get injured

Ski-Rental Problem – Rent vs. Buy



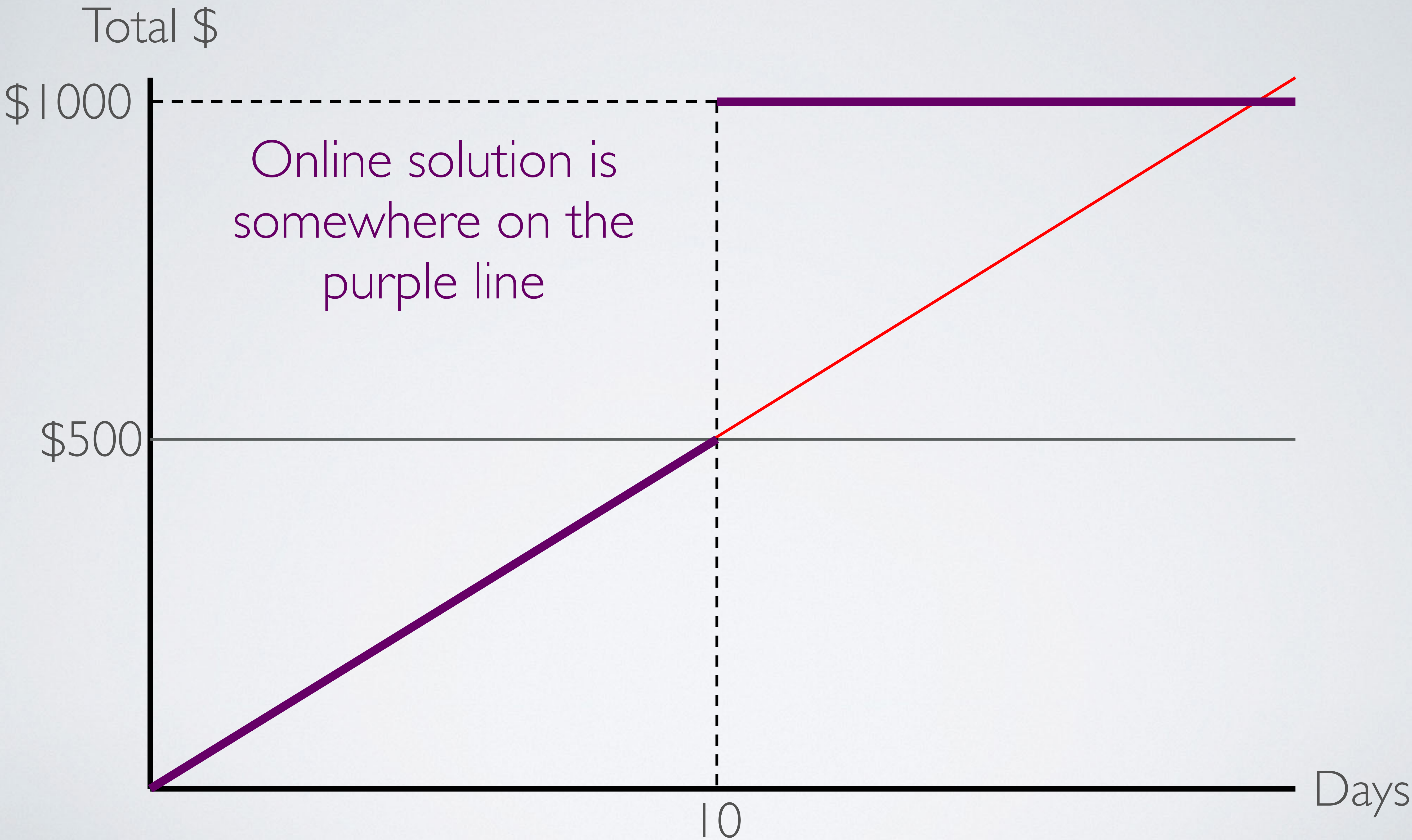
Ski-Rental Problem (Offline)



Ski-Rental Problem (Online)

- We don't know the future, so what can we do?
- Try to get within some constant multiplicative factor of the offline solution!
  - "I want to spend at most **x** times the amount the offline solution would spend"
- Strategy:
  - Rent until total spending equals the cost of buying
  - Then buy if we want to ski some more

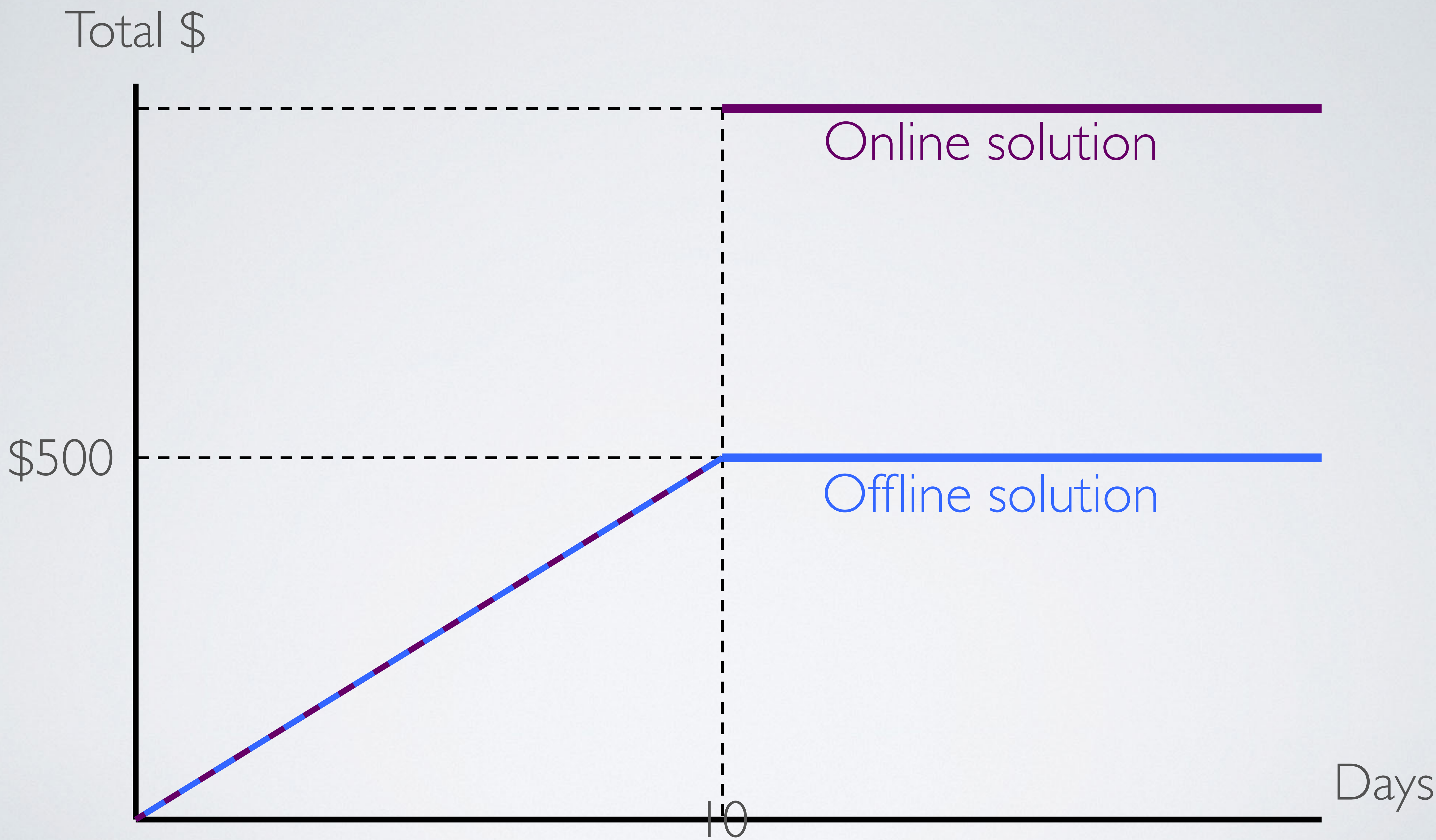
Ski-Rental Problem (Online)



Ski-Rental Problem – Analysis

- How good is this?
  - If we ski 10 days or less, we match the optimal solution!
  - If we ski more than 10 days, we never spend more than twice the offline solution
- This is not the only online solution!

Comparing Solutions



How good is this?

- How do we know that our algorithms are "good", i.e. close to optimal?
- What can we do if we don't even know what the most optimal algorithm is?

Competitive Analysis

- Analyzing an online algorithm by comparing it to an offline counterpart
- *Competitive ratio*: Ratio of performance of an online algorithm to performance of an optimal offline algorithm

$$\text{perf online } \leq c \cdot \text{ perf offline } + \alpha$$

- Our ski-rental solution has a competitive ratio of 2, since we are never more than 2 times as bad as the offline solution
- Our online algorithm is "2-competitive" with the offline solution

More than just skis…

- Refactoring versus working with a poor design
- Dating?

The Experts Problem

- Dating is hard
- You know nothing about dating
- Dating can be reformulated as a series of binary decisions
  - Not "What should I wear?", but
    - Do I wear these shoes? (yes)
    - Should we go at 7? Should we go at 8? (7)
    - Do I wait 15 minutes to text them back? Or 3 hours? (3)
    - Should I buy them flowers? (no)

The Experts Problem: The Scenario

- You know nothing, so you should ask for help
- You know **n** experts who can give you advice before you make each decision (but you don't know if it's good)

The Experts Problem

- Rules
  - If you make the right decision, you gain nothing
  - If you make the wrong decision, you get 1 unit of embarrassment
  - Total embarrassment = number of mistakes

- Goal: Minimize total embarrassment (relative to what the best expert would've gotten)

The Experts Problem (Offline)

- Offline:
  - We know the best expert
  - We only listen to them
  - Whatever successes and mistakes they have, we have
- Online:
  - We don’t know the best expert

The Experts Problem (Online)

- Assign every expert a weight of 1, for total weight of **W = n** across all experts
- Repeat for every decision:
  - Ask every expert for their advice
  - *Weight* their advice and decide by majority vote
  - After the outcome is known, take every expert who gave bad advice and cut their weight in half, regardless of whether your bet was good or bad

Lecture Activity 1

Fill in the blank weights on your sheet!

Round 1: Should I buy them flowers?
What do the experts say?

- Expert 1, 2, 3 say no
- Expert 4, 5 say yes

Correct Answer? **Yes!**

Round 2: Should I show up fashionably late?
What do the experts say?

- Expert 3, 5 say no
- Expert 1, 2, 4 say yes

Correct Answer? **No!**

2 Mins….

Lecture Activity 1

Fill in the blank weights on your sheet!

Round 1: Should I buy them flowers?
What do the experts say?
- Expert 1, 2, 3 say no
- Expert 4, 5 say yes

Correct Answer? **Yes!**

Round 2: Should I show up fashionably late?
What do the experts say?
- Expert 3, 5 say no
- Expert 1, 2, 4 say yes

Correct Answer? **No!**

1 Min….

Lecture Activity 1

Fill in the blank weights on your sheet!

Round 1: Should I buy them flowers?
What do the experts say?

- Expert 1, 2, 3 say no
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Correct Answer? **Yes!**

Round 2: Should I show up fashionably late?
What do the experts say?

- Expert 3, 5 say no
- Expert 1, 2, 4 say yes

Correct Answer? **No!**

0 Mins….

Lecture Activity 1 Answers

Updating the weights

Weights	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5
Initial	1	1	1	1	1
Round 1	0.5	0.5	0.5	1	1
Round 2	0.25	0.25	0.5	0.5	1

- Let's see how we can make a decision in the third round!
- Remember, sum the weights of the experts of both options and pick the majority value!

Lecture Activity 2

Which decision should we make?

Round 3: Should I order the clams and garlic?
What do the experts say?

- Expert 1, 2, 3 say yes
- Expert 4, 5 say no

1 Min….

Lecture Activity 2

Which decision should we make?

Round 3: Should I order the clams and garlic?
What do the experts say?

- Expert 1, 2, 3 say yes
- Expert 4, 5 say no

0 Mins….

Lecture Activity 2 Answer

Which decision should we make?

Round 3: Should I order the clams and garlic?
What do the experts say?

- Expert 1, 2, 3 say yes
- Expert 4, 5 say no

Majority Decision
Yes sum: 0.25 + 0.25 + 0.5 = 1
No sum: 0.5 + 1 = 1.5
Majority answer is **No**, so we don't eat clams and garlic! Good choice…

- How good is our algorithm?

Multiplicative Weights Algorithm - Analysis

- To analyze how good this is, we need to relate the number of mistakes we make (**m**) to the number of mistakes the best expert makes (**b**)
- How can we do that? Use the weights!
- Let **W** represent the sum of the weights across the **n** experts at an arbitrary point in the algorithm

Experts Algorithm - Analysis

- Look at the total weight assigned to the experts
- When the best expert makes the wrong decision…
  - We cut their weight in half
  - They started out with a weight of 1

$$\left(\frac{1}{2}\right)^b \leq W$$

Experts Algorithm - Analysis

- Look at the total weight assigned to the experts
- When we made the wrong decision…
  - At least ½ weight was placed on the wrong decision
  - We will cut at least ¼ of **W**, so we will reduce the total weight to at most ¾ of W
  - Since we gave the experts n total weight at the start:

$$W \leq n \left(\frac{3}{4} \right)^m$$

Experts Algorithm - Analysis

$$W \leq n\left(\frac{3}{4}\right)^m$$

$$\left(\frac{1}{2}\right)^b \leq W$$

$$\left(\frac{1}{2}\right)^b \leq W \leq n\left(\frac{3}{4}\right)^m$$

Experts Algorithm - Analysis

$$\left(\frac{1}{2}\right)^b \leq W \leq n\left(\frac{3}{4}\right)^m$$

$$\left(\frac{1}{2^b}\right) \leq W \leq n\left(\frac{3}{4}\right)^m$$

$$\left(\frac{1}{2^b}\right) \leq n\left(\frac{3}{4}\right)^m$$

$$-b \leq \log_2 n + m \log_2\left(\frac{3}{4}\right)$$

$$-b - \log_2 n \leq m \log_2\left(\frac{3}{4}\right)$$

$$b + \log_2 n \geq m \log_2\left(\frac{4}{3}\right)$$

$$\frac{(b + \log_2 n)}{\log_2\left(\frac{4}{3}\right)} \geq m$$

So the number of mistakes we make, ***m***, is at most 2.41 times the number of mistakes the best expert makes, ***b***, plus some change

We want THE BEST

- How to find the best…
    - apartment?
    - deal for a ticket?
    - class to take?
    - partner?
- How can we know that they're the best?
- How much effort are we willing to spend to find the best one?

The {Best Choice, Dating} Problem

- Also known as the secretary problem
- There are **n** people we are interested in, and we want to end up dating the best one
- Assumptions:
  - People are consistently comparable, and *score*(***a***) != *score*(***b***) for arbitrary people ***a*** and ***b***
  - You don't know anyone's score until you've gone on at least one date with them
  - Can only date one person at a time (serial monogamy)
  - Anyone you ask to stay with you will agree to do so

Dating Problem

- What's the offline solution?
  - If you already know everyone's score, just pick the best person
- A naïve online solution?
  - Try going out with everyone to assign them scores, and ask the best person to take you back
- Problems:
  - Takes a lot of time / money, depending on **n**
  - Assumes that they will take you back

Dating Problem

- Two main constraints:
  - You can't look ahead into the future
  - There's no "undo" – if you let someone go, chances are they'll be taken by the time you ask for them back
- In other words, the problem is: Do I reject the current possibility in hopes of landing something better if I keep looking, or do I stick with what I have?

Dating Problem

- Solution:
  - Pick a random ordering of the **n** people
  - Go out with the first **k** people.
  - No matter how the dates go, reject them (calibration of expectations)
  - After these **k** dates, pick the first person that's ***better than everyone we've seen so far,*** and stick with them – they're probably the best candidate

Dating Problem - Analysis

- What value of **k** maximises our chances of ending up with the best person?
- 3 Cases to consider:
  - What if the best person is in the first **k**?
    - We end up alone. Oops.
  - What if the person that we pick isn't actually the best?
    - Oh well, we live in blissful ignorance
  - Otherwise, we successfully pick the best person!

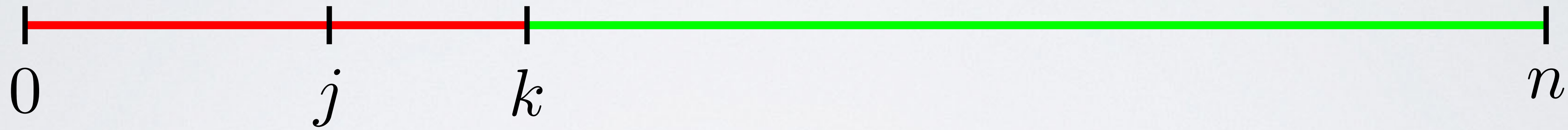
Dating Problem - Analysis

- Consider the candidate at position **j**
- Let's first consider the probability that the algorithm pairs us with this candidate, given a value of **k**

$$P_{choose}(k, j) = \begin{cases} 0 & \text{if } j \leq k, \\ \dfrac{k}{j-1} & \text{otherwise} \end{cases}$$

Dating Problem - Analysis

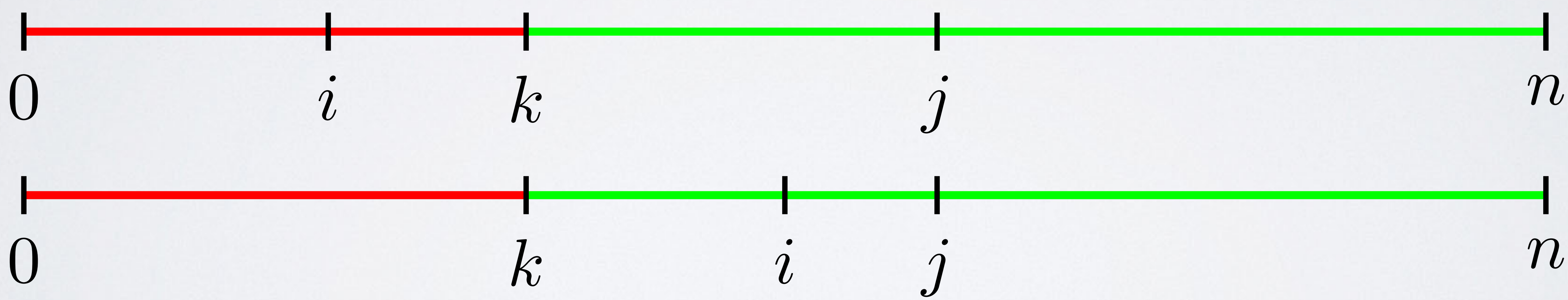
- Consider the candidate at position **j**
- Case 1



$$P_{choose}(k, j) = \begin{cases} 0 & \text{if } j \leq k, \\ \dfrac{k}{j-1} & \text{otherwise} \end{cases}$$

Dating Problem - Analysis

- Case 2:
  - There exists some person at position *i* who has the highest score we've seen so far by the time we're considering the jth person



$$P_{choose}(k,j) = \begin{cases} 0 & \text{if } j \leq k, \\ \dfrac{k}{j-1} & \text{otherwise} \end{cases}$$

Dating Problem - Analysis

- The probability that the **j**th person actually ***is*** the best is 1 / n)
- For a given **k**, the probability that we end up with the best person, P_{best}, is the sum of the conditional probabilities for each valid value of **j**

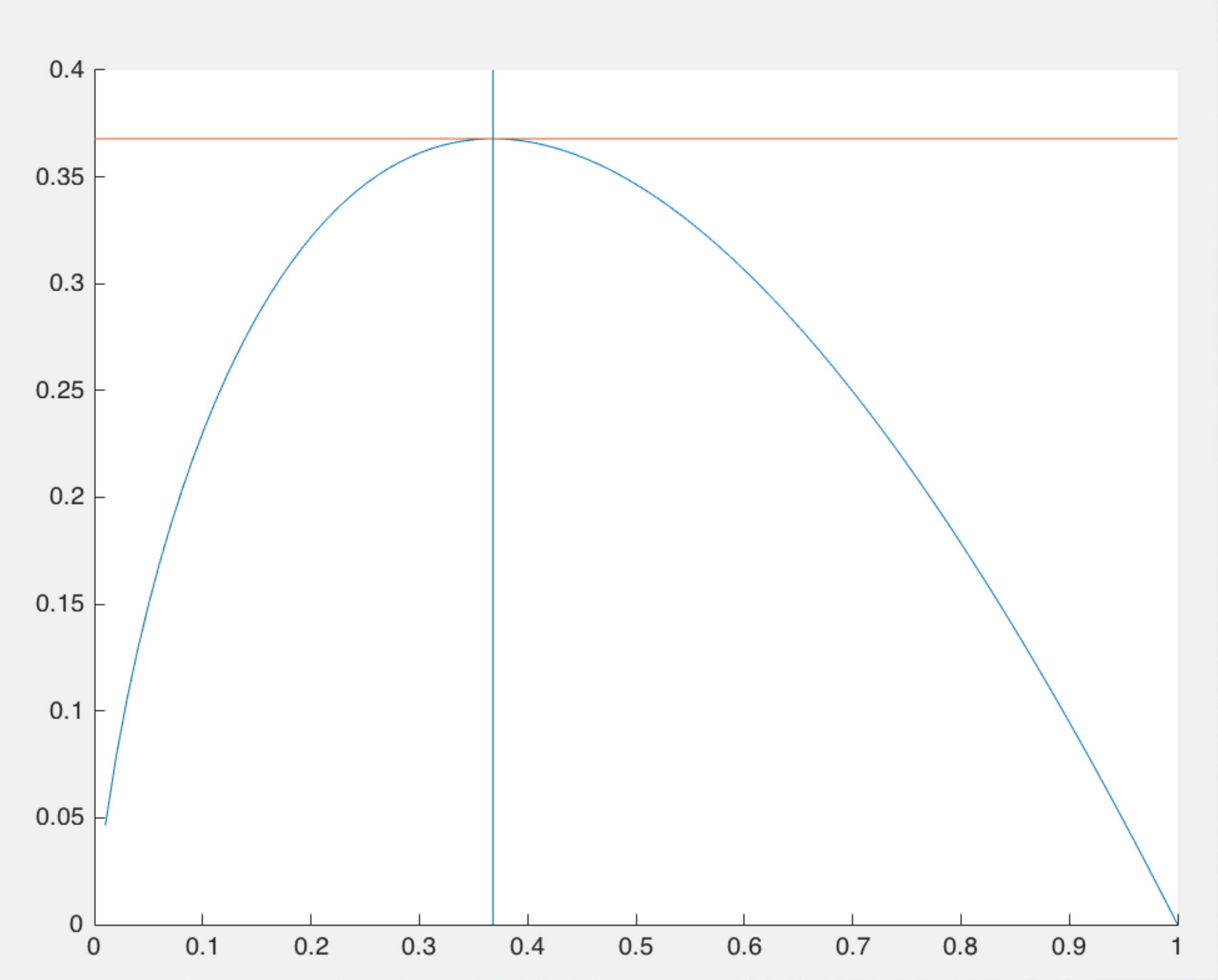
$$P_{best}(k) = \sum_{j=k+1}^{n} \left(\frac{k}{j-1}\right)\left(\frac{1}{n}\right)$$

Dating Problem - Analysis

$$P_{best}(k) = \sum_{j=k+1}^{n} \left(\frac{k}{j-1}\right)\left(\frac{1}{n}\right)$$

$$= \left(\frac{k}{n}\right) \sum_{j=k+1}^{n} \left(\frac{1}{j-1}\right)$$

$$\lim_{n\to\infty} \approx -\left(\frac{k}{n}\right) \ln\left(\frac{k}{n}\right)$$



In the above graph, what's the k/n value that maximizes p_{best}?
And what's the maximum value?

Dating Problem - Analysis

- **1/e**, for both the maximum value of P_{best} and the maximizing input for k/n

$$\frac{1}{e} = \frac{k}{n} \implies k = \frac{n}{e}$$

- So, with 1/e = 36.79% probability, if your strategy is to date the first person better than everyone in the first 36.79% of dates, you'll end up with the best person!

Dating Problem - Improvements

- **1/e** probability of not ending up with anyone :(
  - Strategy: Be desperate
    - Pick the last person, if you get that far
    - With probability 1/e, we pick the last person who will have, on average, rank n/2, so we'll probably be ok
  - Strategy: Gradually lower expectations
    - Pick a series of timesteps, t_0, t_1, t_2, t_k…
    - Reject the first t0 dates as before
    - Look for the best person we've seen so far between dates t_0 and t_1
    - If we find them, great!
    - Otherwise, between dating the (t_1+1)th and t_2th people, look for either the first or the second best we haven't yet dated
    - Repeat the above, gradually accepting a larger "pool"
    - We'll probably do better than the "be desperate" strategy, though by how much is hard to say without hardcore math

Recap

- An ***online algorithm*** is an algorithm where input is fed to you piece by piece, which makes writing a fast and optimal algorithm much more difficult
- Competitive analysis frames an online algorithm's efficiency in terms of an offline solution

CS Applications

- CPUs and memory caches (CS33, CS157)
  - Intel pays major $$$ for good caching strategies
- Artificial intelligence (CS141)
  - Heuristics, search, genetic algorithms
- Machine learning (CS142)
- Statistics

More Applications (continued)

- Economics
  - Stocks and trading
  - Game theory
  - Gambling
- Biology (featuring 2 authors of our textbook, Papadimitriou and Varizani)
  - https://www.quantamagazine.org/20140618-the-game-theory-of-life/
  - Our textbook: Dasgupta, Papadimitriou, and Varizani
  - Evolution as a balance between fitness and diversity, given an unknown future

Dealing with Hard Problems

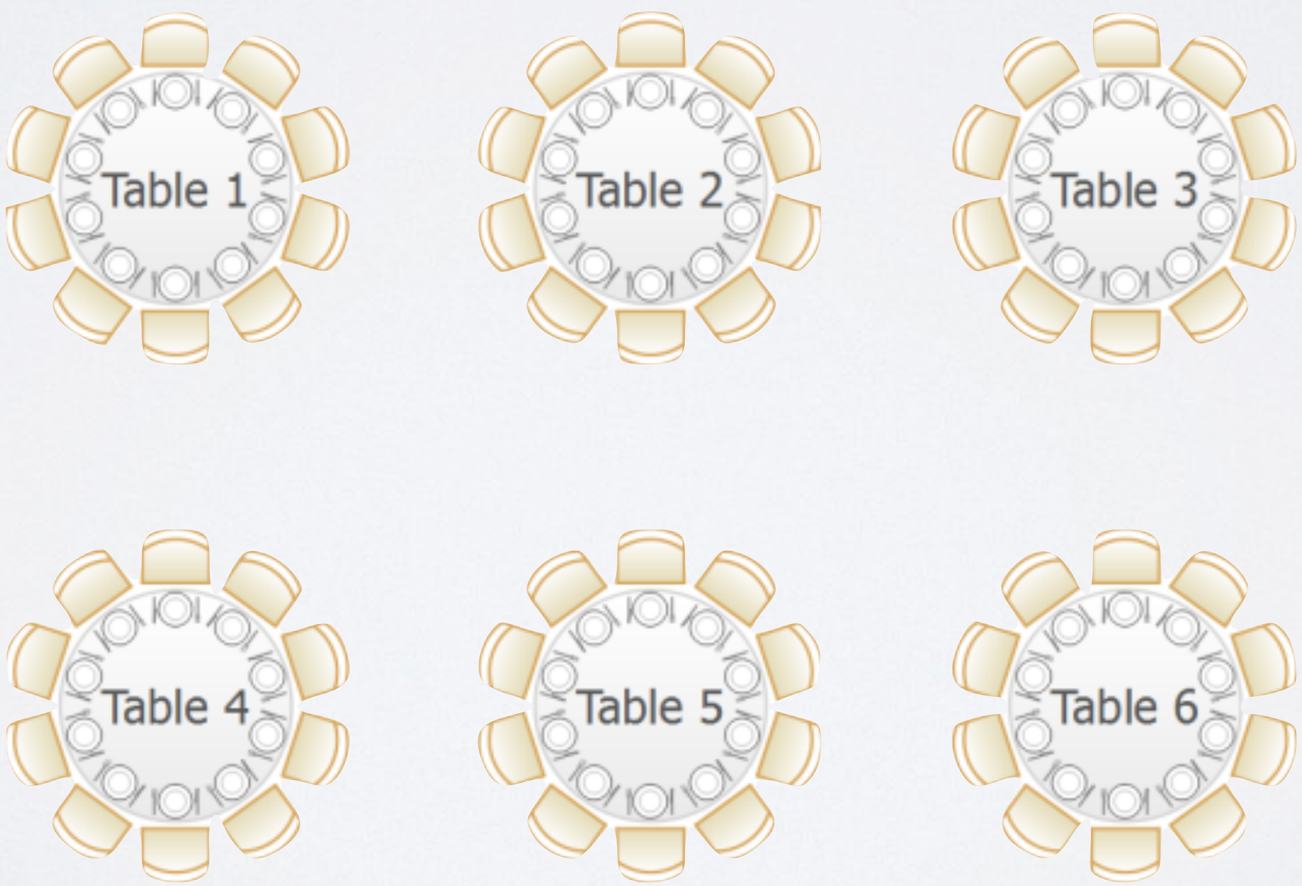
CS16: Introduction to Data Structures & Algorithms
Spring 2020

Outline

- Seating Arrangements
- Problem hardness
- P, NP, NP-Complete, NP-Hard
- Dealing with hard problems
  - Problem translation
  - Genetic Algorithms
  - Approximations
- Travling Salesman Problem

Seating Arrangement Problem

- Your dating algorithms worked!
- You need to plan the seating arrangements for a wedding



Seating Arrangement Problem

- Constraints / goals
  - **k** tables
  - **n** people
  - Avoid antagonistic pairs (exes, rivals, etc) sitting at the same table
  - Maximise overall happiness

Quantifications of Pair-wise Happiness

- Assume each pair of people (A, B) has an associated 'compatibility score'
  - for **friends** comp(A, B) = **10**
  - for **couples** comp(A, B) = **50**
  - for **antagonistic pairs** comp(A, B) = **-500**
- These values are known ahead of time

Quantifications of Table-wise Happiness

- Sum all the compatibility scores for each pair at the table

$$H\left(table\right) = \sum_{pair \in table} comp(pair)$$

Quantification of Total Happiness

- Utilitarian Approach:

$$Total\_H_{utilitarian} = \sum_{t \in tables} H(table)$$

- Egalitarian Approach:

$$Total\_H_{egalitarian} = \min_{t \in tables} H(t)$$

- Many more options!

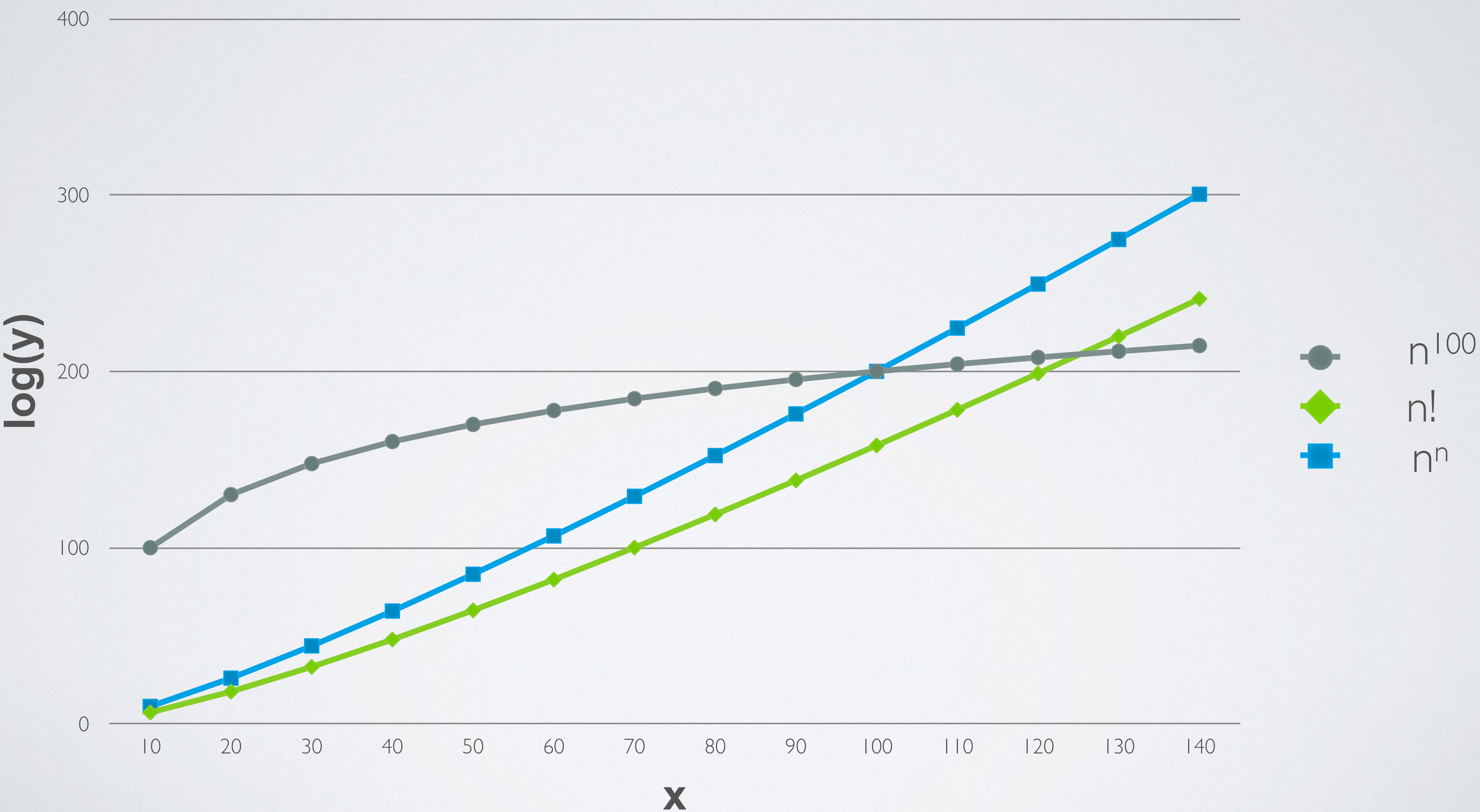
This seems hard

- Could we just try permutations and comparing scores?
- With 60 people, 60! permutations to test
  - 8.32×10^{81}
  - ouch
- This doesn't necessarily mean that the problem *is* hard, however

Defining Problem Hardness

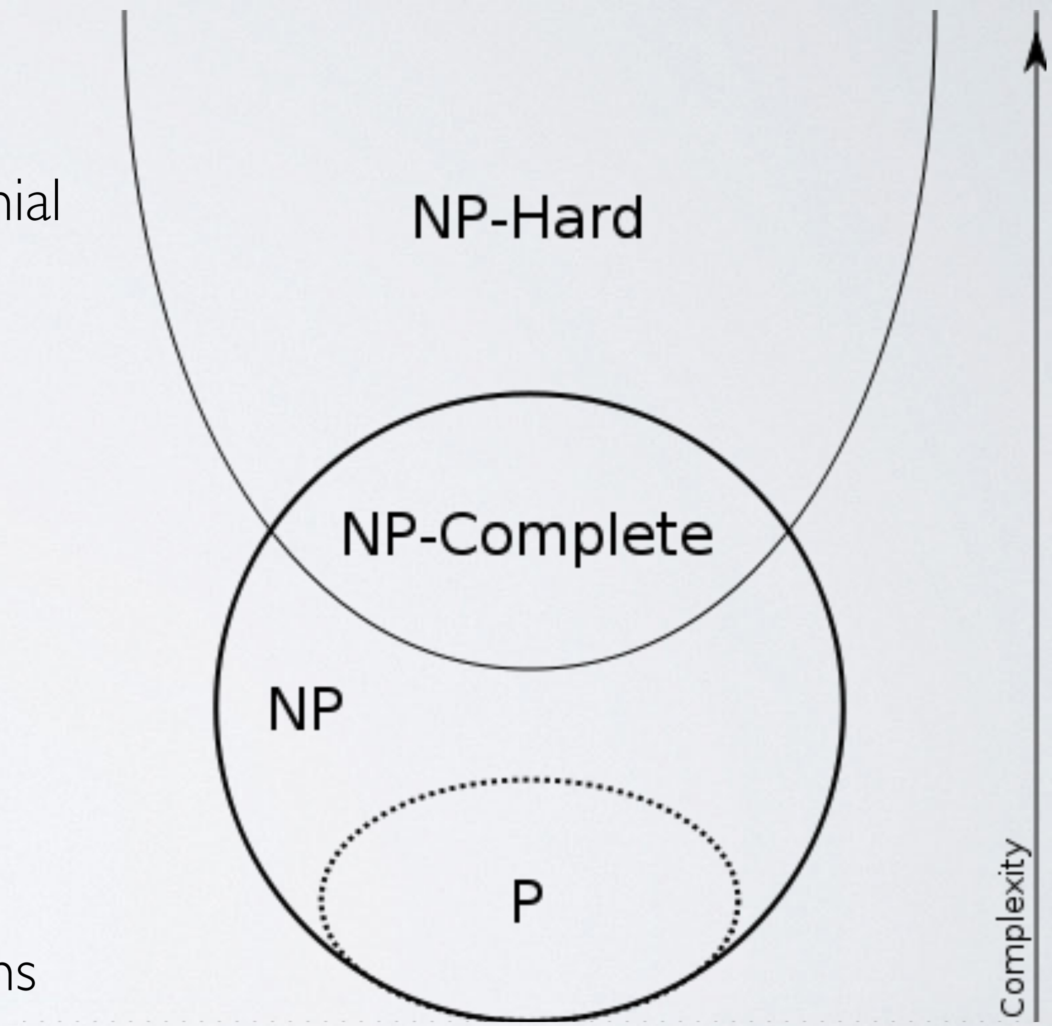
- Hardness of problem is defined by the runtime of the best solution
  - A bad sorting algorithm *could* be $O(n!)$, but sorting in general isn't considered hard, because we have fast algorithms to solve it
- Polynomial Runtimes
  - $O(n)$, $O(n^2)$, $O(n^{500})$
  - Problems with these solutions are **tractable**
- Super-Polynomial Runtimes
  - $O(n!)$, $O(2^n)$, $O(n^n)$
  - Problems with these solutions are **intractable**

Exponential vs. Polynomial Growth Rates



Categories of Hardness

- NP
    - The set of problems for which we can verify the correctness of a solution in polynomial time
- P
    - A subset of NP, where the problem is solvable in polynomial time
- NP-Complete
    - "The hardest problems in NP"
    - Solution is checkable in polynomial time
    - not known whether there exist any polynomial time algorithms to solve them
- NP-Hard
    - Problems that are "at least as hard as the hardest problems in NP"
    - Don't necessarily have solutions that are checkable in polynomial time



Back to our seating arrangement

- To get an intuition as to how hard our problem is, let's see if we can convert it into a problem that has already been proven to be in NP, P, NP-Complete, or NP-Hard
- But… where to start?

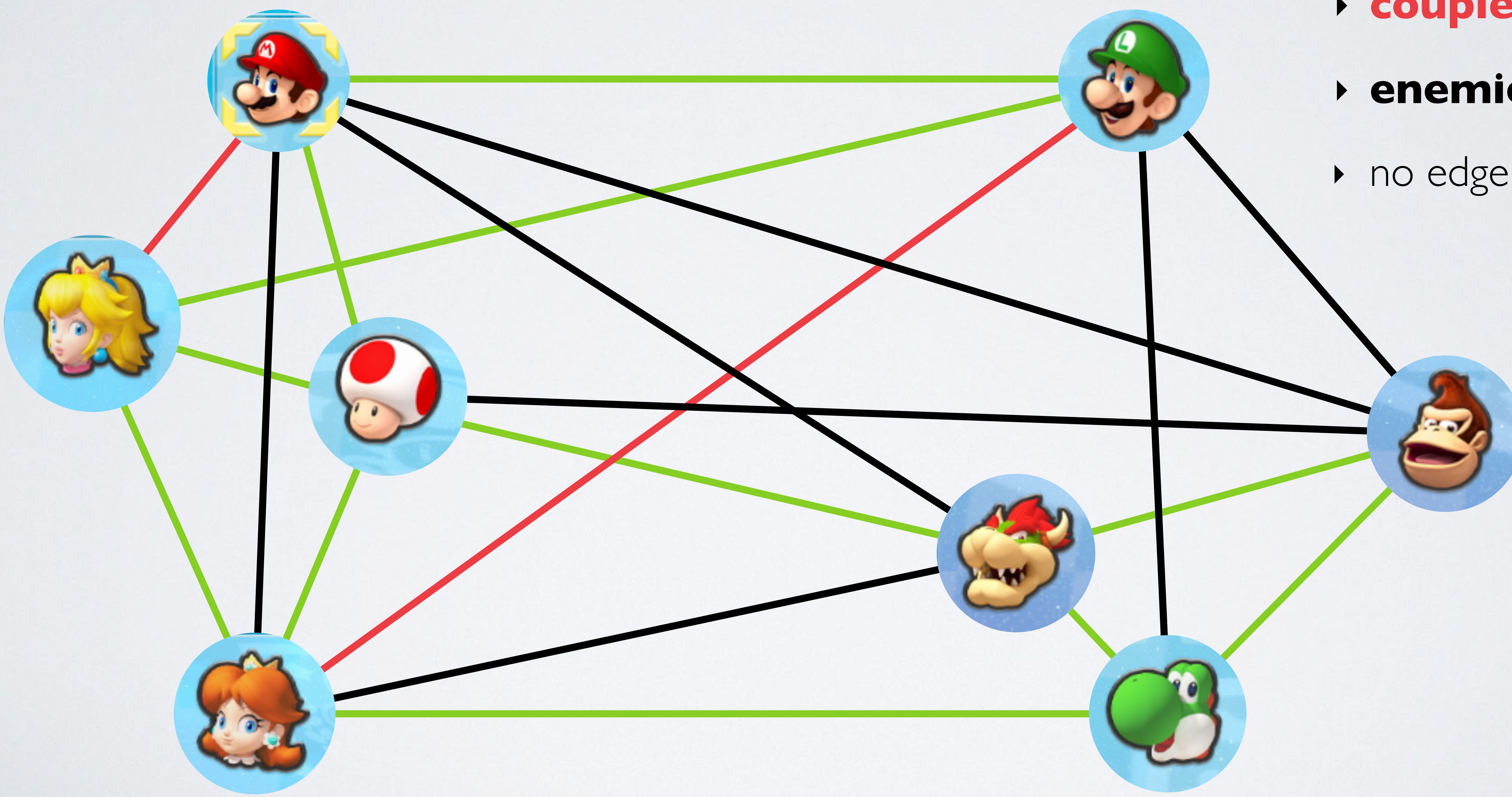
Constraint Relaxation

- See if you can solve an 'easier' version of the problem, by removing some of the properties that make the problem hard
- In real life
  - "what would you do if you could not fail?"
  - "which job would you take if they all paid equally?"

Let’s avoid disaster

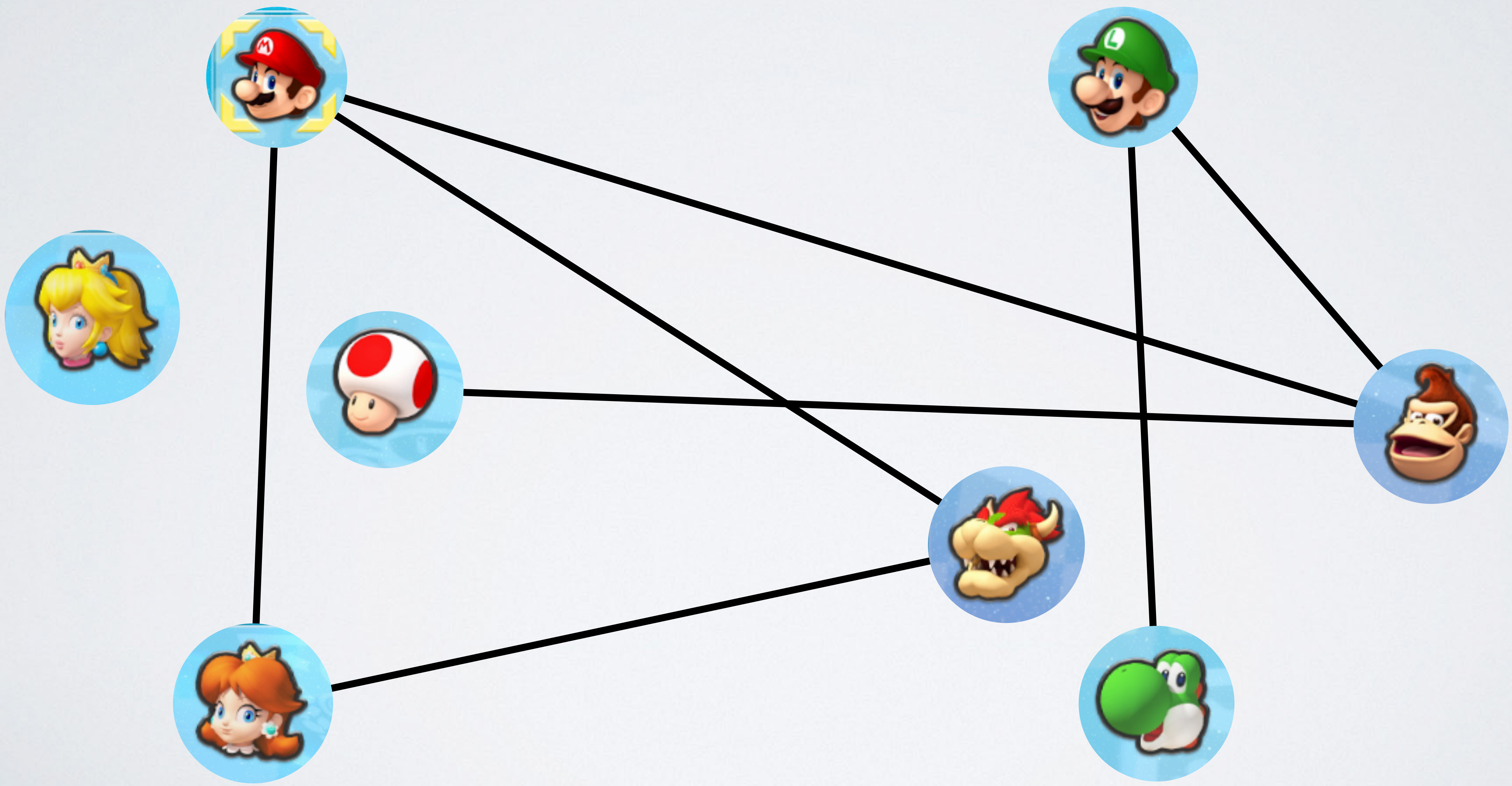
- Constraints / goals
  - # of tables
  - # of people
  - Avoid antagonistic pairs (exes, rivals, etc)
  - ~~Maximise overall happiness~~
- Hopefully, having no tables with antagonistic pairs will put in the right direction for maximising overall happiness

Relationships as a graph



- edge key:
  - **friends**
  - **couple**
  - **enemies**
  - no edge = no prefs

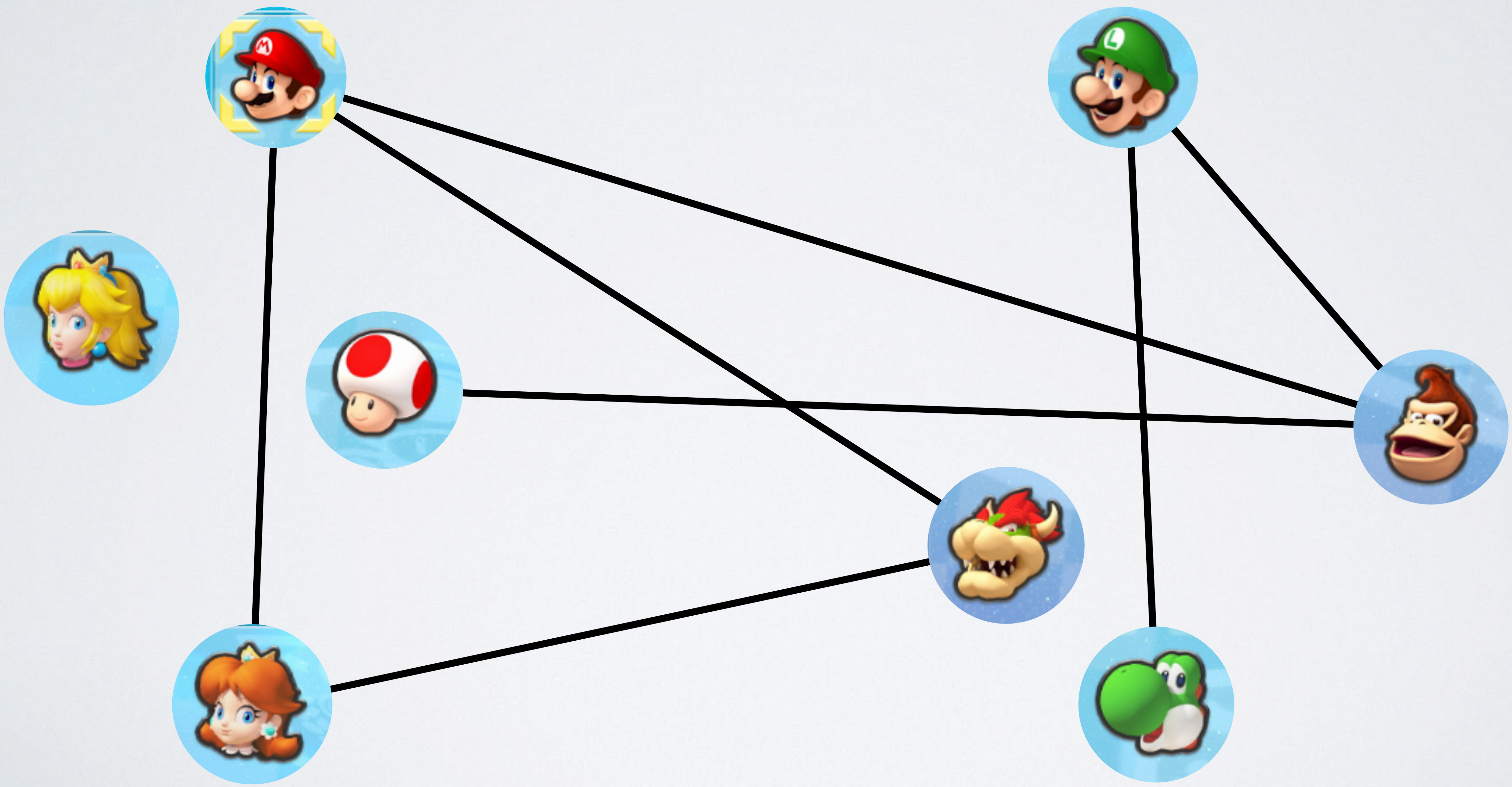
An Antagonism graph



Translating the problem

- Now, we have these antagonistic relationships represented as a graph!
- Question is no longer:
  - Can we avoid antagonistic pairs (exes, rivals, etc) sitting at the same table, given **n** people and **k** tables?
- Instead:
  - Use colours to represent different tables, so:
  - Could we assign 1 of **k** colours to each node in the antagonism graph, such that no two nodes that share an edge have the same colour?

An Antagonism graph



Lecture Activity 3

Try out the Graph k-colouring problem!

2 Mins….

Lecture Activity 3

Try out the Graph k-colouring problem!

1 Min….

Lecture Activity 3

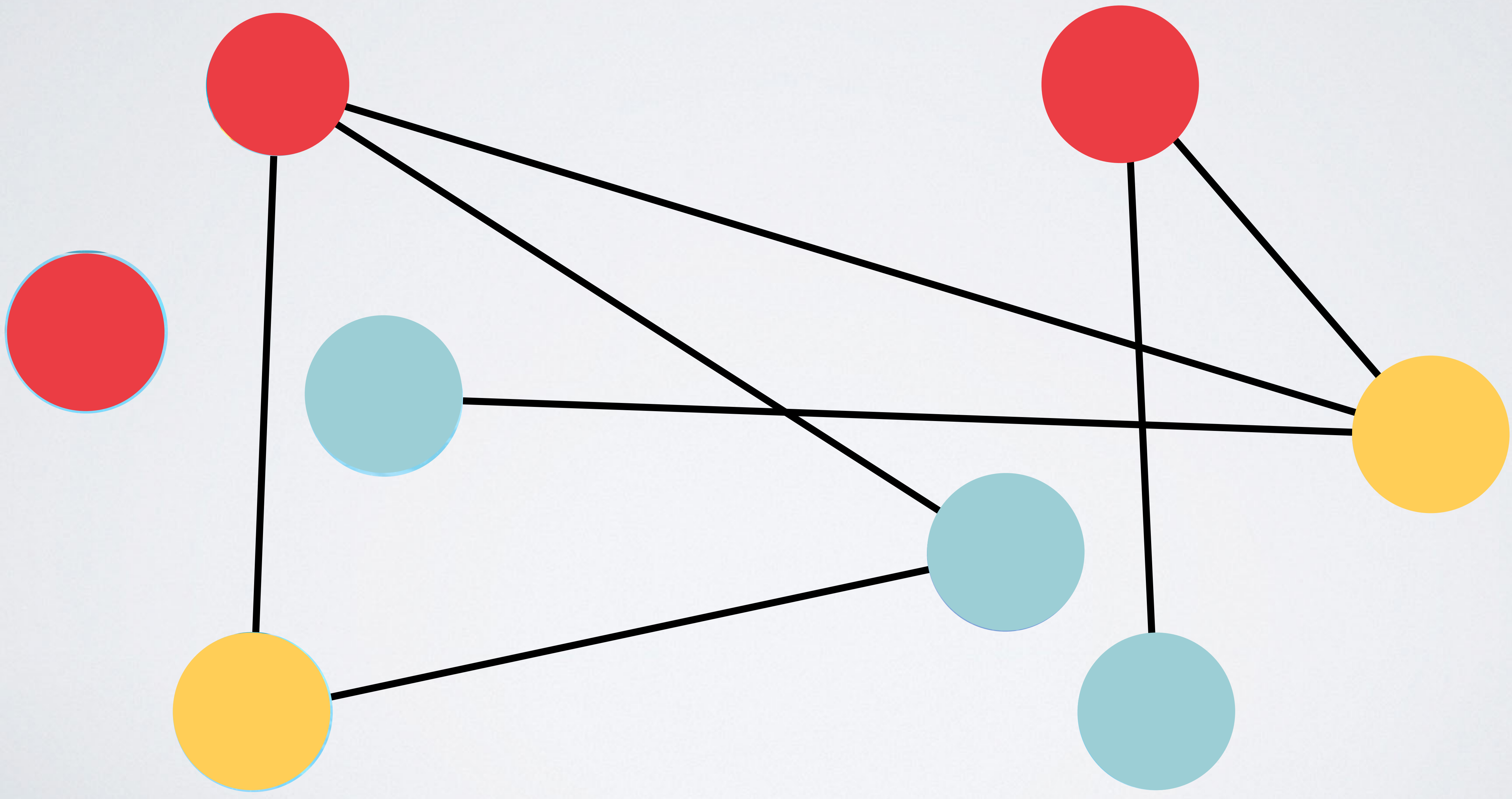
Try out the Graph k-colouring problem!

0 Mins….

Lecture Activity 3 Answers

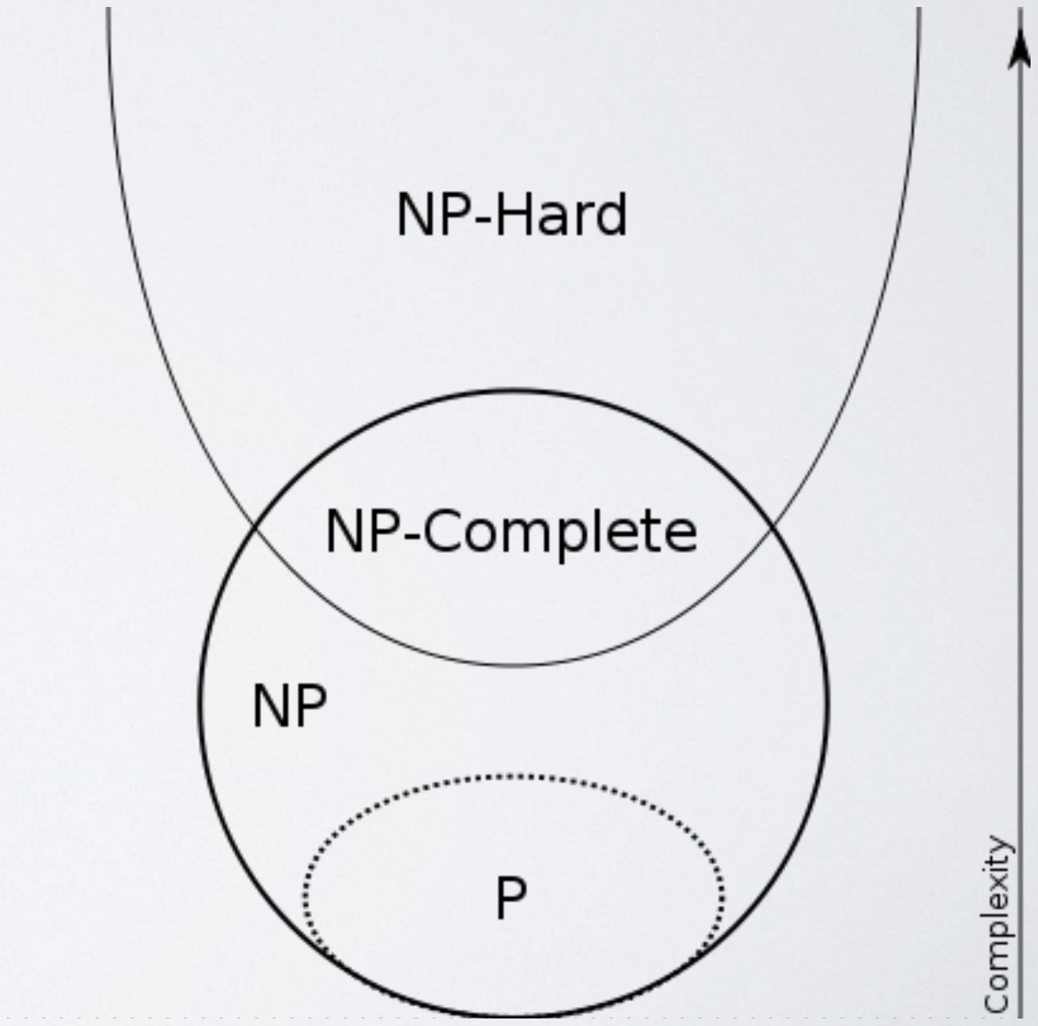
Answers!

Graph colouring example



Graph k-colouring

- Generally, the problem of determining whether nodes in a graph can be coloured using up to **k** separate colours, such that no two adjacent vertices share a colour
- This is NP-Complete!
- And thus, even this much easier version of the problem is **very hard**



Are we screwed?

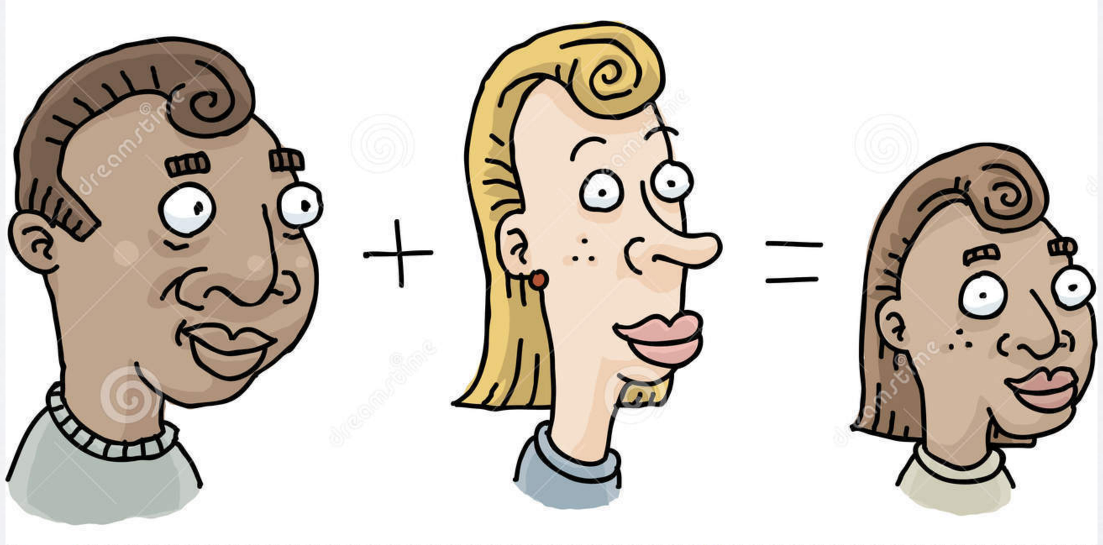
- The best algorithms to solve the graph k-colorability problem take $O(2.445^n)$ time and space
- With 60 guests, 2.445^{60} = ~450 billion
  - which isn't *that* bad
  - Modern computers can handle ~3 billion 'operations' / sec, so this would take more than a couple minutes, probably less than 15
- But we've still only avoided the worst case!

Genetic Algorithms

- A form of 'guess and check', using a number of possible solutions to a problem
- Inspired the process of evolution

Biology Review

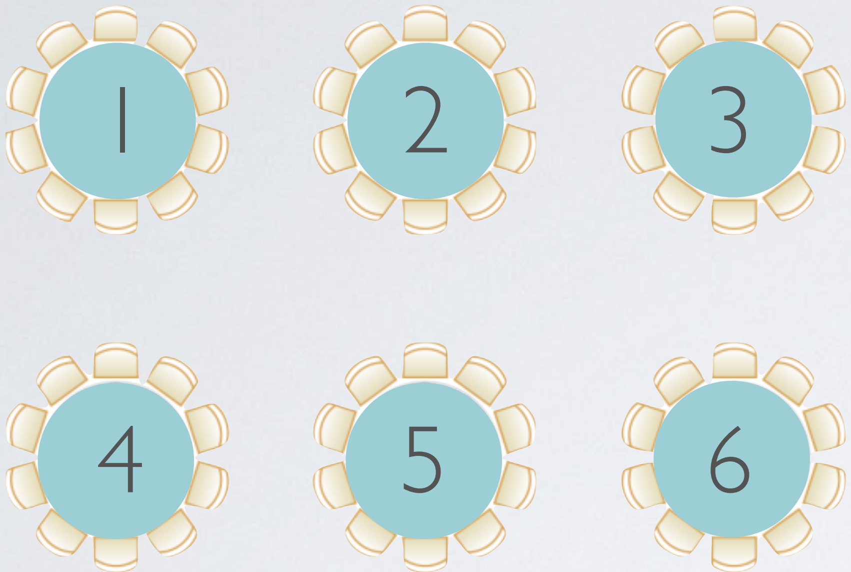
- All organisms are made up of **genes**, where genes (or a combination many genes) interact to produce our **phenotype**, the expression of those genes
- We are all a combination of a mix of our parents genes, and some random mutations



Evolution via Sexual Reproduction, broadly

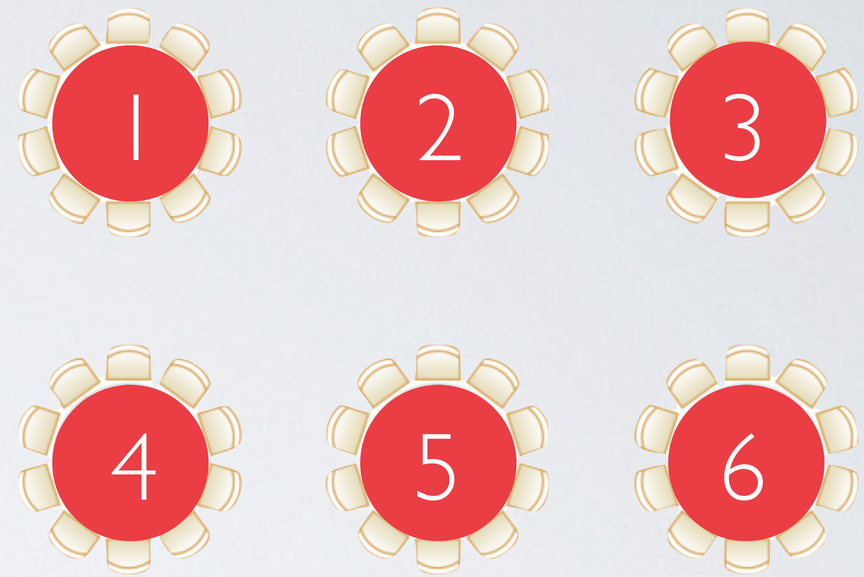
- There exist an initial population of organisms within a species
- The ‘sexually fit’ organisms reproduce
  - Take some genes of parent A, some of parent B
  - add some random noise
  - this new collection of genes is a new specimen, AB’
- Older + less fit parts of populations die off, leaving the survivors to repeat the reproduction process

Solution Mating

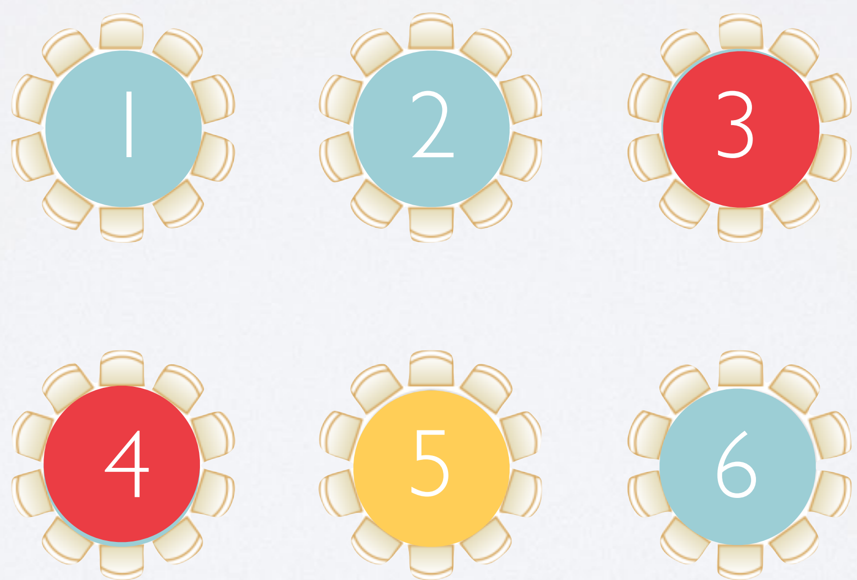


Total_H = 300

+



Total_H = 325



Total_H = 400

High-Level Genetic Algorithm Pseudocode

```
function geneticAlgo(opt_seed_sols):
   solution_set = opt_seed_sols or || randomly generated initial population of solutions
   init_size = size(solution_set, threshold, time_limit)

   while True:
        new_gen = []
        for some number of iterations:
           A, B = 2 solutions from solution_set, drawn at random
           AB' = a new solution that combines properties of A and B
           randomlyMutate(AB')
           new_gen.append(AB')
        solution_set.addAll(new_gen)
        rank solutions in solution_set based on 'fitness'
        remove all but init_size many best solutions from solution_set
        if best(solution_set) > threshold or time_limit has passed:
           break
  return highest ranking solution from solution_set
```

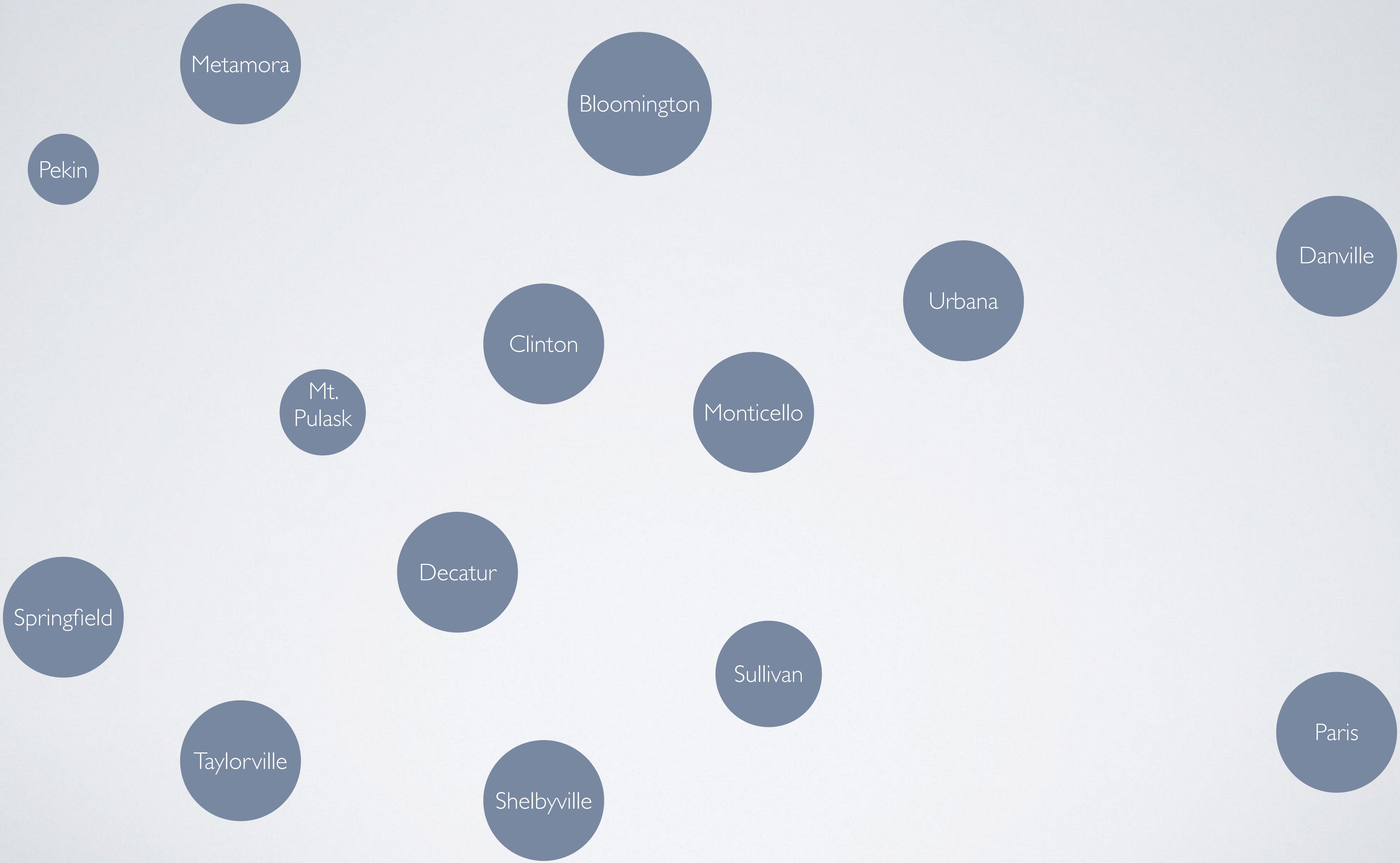
Genetic Algorithms

- If seeded with 'good' solutions for the initial population of solutions, output is guaranteed to be at least as good as the best of the initial solutions
- Can come up with unexpected solutions
- Tend to do really well!

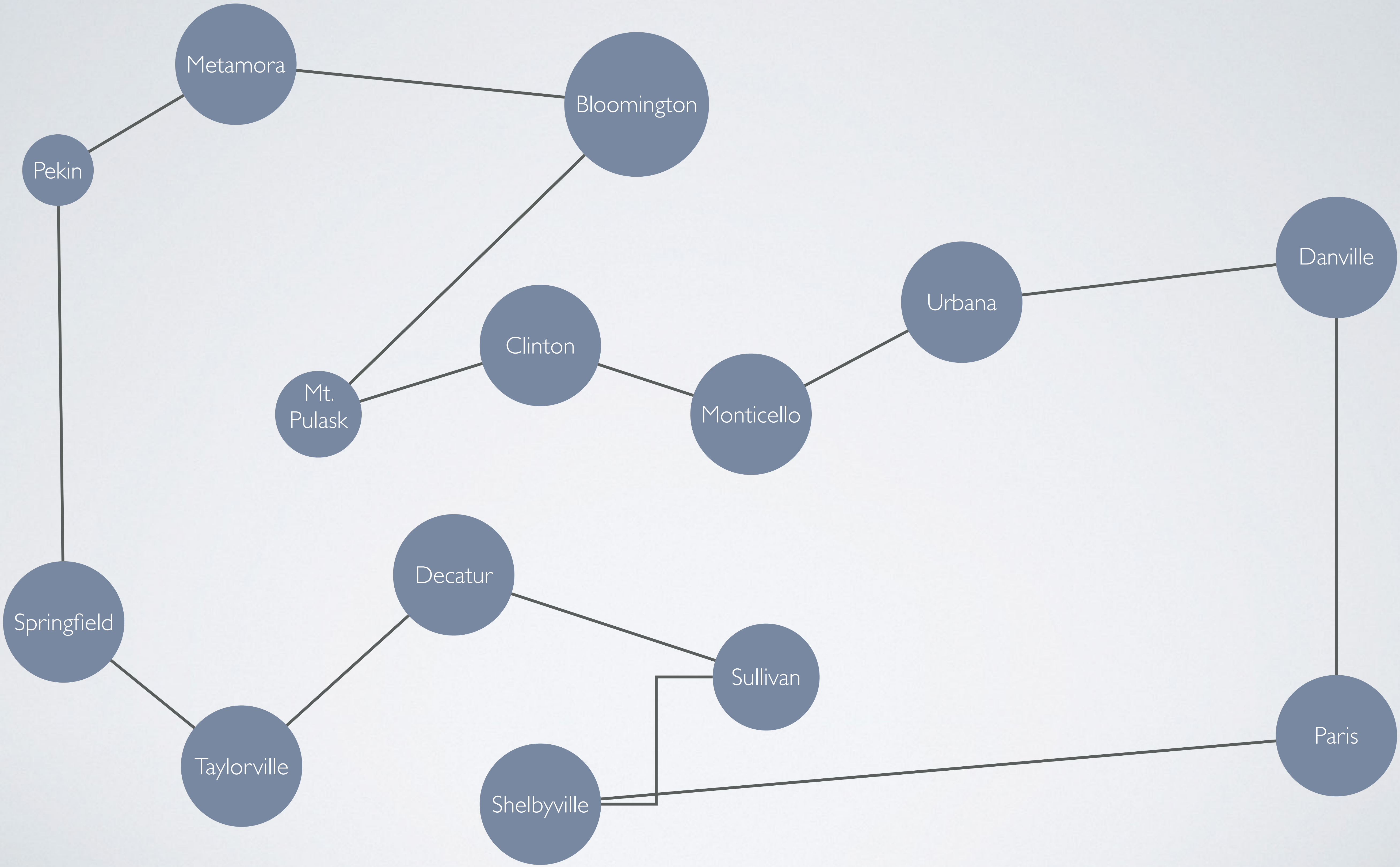
Honeymooning

- Also known as the Traveling Salesman Problem
- TSP, defined: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"

Cities (not to scale):



Best route:


Metamora
Bloomington
Pekin
Danville
Urbana
Clinton
Mt.
Pulask
Monticello
Decatur
Springfield
Sullivan
Paris
Taylorville
Shelbyville

TSP Hardness

- Given a graph with **n** nodes
  - we *could* exhaustively try O(n!) possible city-orderings
  - But let's see if we can do any better
- Finding the most optimal route is NP-Hard :(
- Held-Karl algorithm solves it in $O(n^2 \times 2^n)$

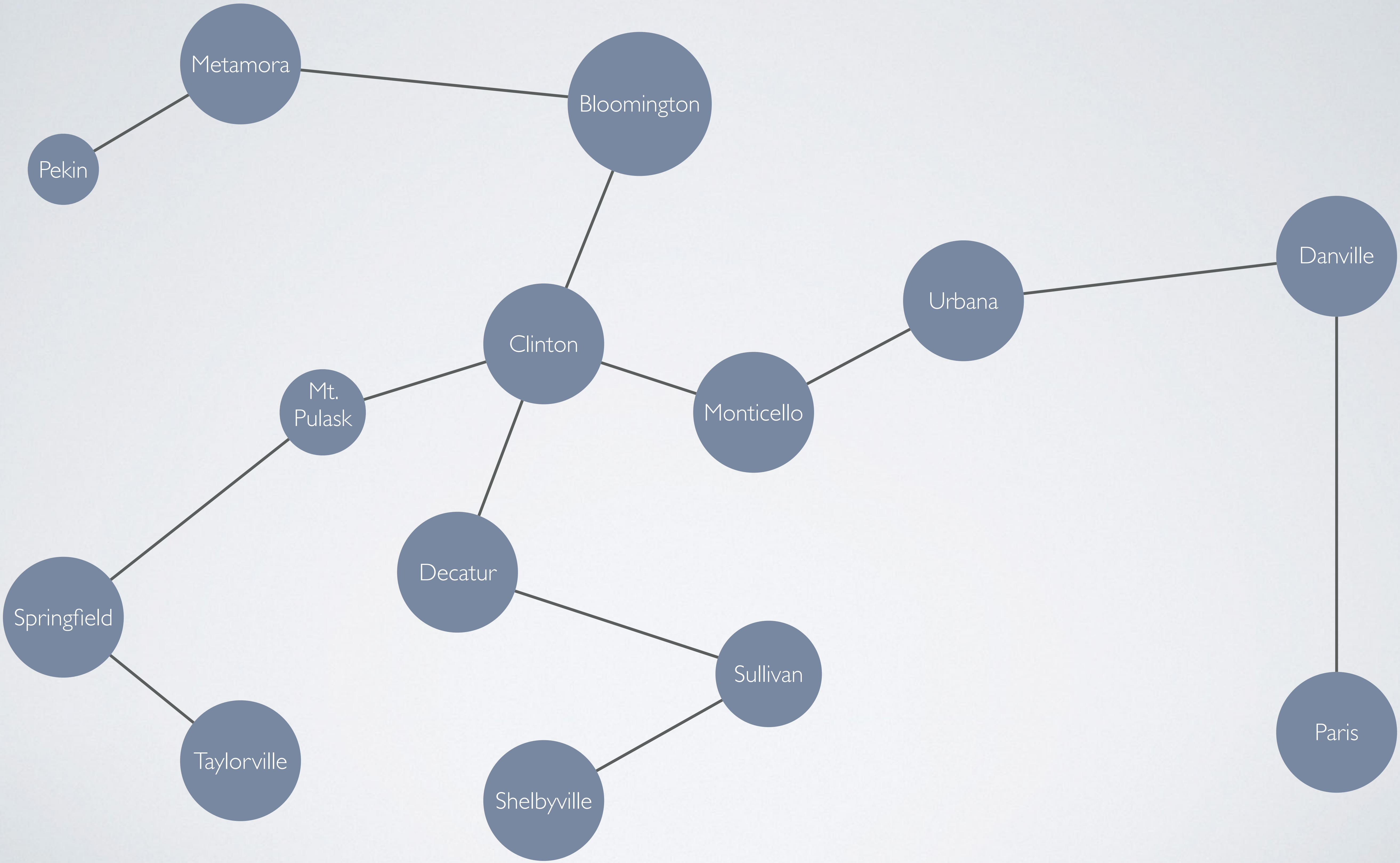
But we’re not totally screwed!

- Again, relaxing constraints…
- What if we were
  - allowed to visit a city more than once, and
  - allowed to retrace your steps for free?
- Sounds like the problem reduces to connecting alls the cities as cheaply as possible - do we know how to solve this problem?

MSTs as a starting point to approximate TSP

- This is very easy!
- Provides a lower bound for the real solution
  - a solution with free backtracking can't possibly be worse than a solution that has to follow all the original rules
  - If we find a solution to the original problem, can use the MST as a comparison for how close we might be
    - If an MST for some graph has total 100 mile distance, but a given solution has total distance of 110, we are at most 10% longer than the best solution

MST of cities:



Best route vs. MST