Minimum Spanning Trees:
Prim-Jarnik & Kruskal

CS16: Introduction to Data Structures & Algorithms
Spring 2019
Outline

- Minimum Spanning Trees
  - Analysis
  - Proof of Correctnessness
- Prim-Jarnik Algorithm
  - Analysis
  - Proof of Correctness
- Kruskal’s Algorithm
  - Union-Find
  - Analysis
  - Proof of Correctness
Spanning Trees

- A **spanning tree** of a graph is
  - subset of edges that form a tree that spans every vertex
Minimum Spanning Trees

- A **minimum spanning tree** (MST) is a spanning tree with minimum total edge weight
Applications

- Networks
  - electric
  - computer
  - water
  - transportation
- Computer vision
  - Facial recognition
  - Handwriting recognition
- **Image segmentation**
- Low-density parity check codes (LDPC)
Minimum Spanning Tree Algos

- Prim-Jarník Algorithm

“Shortest Connection Networks And Some Generalizations” by R. C. Prim
(Manuscript received May 8, 1957)

The basic problem considered is that of interconnecting a given set of terminals with a shortest possible network of direct links. Simple and practical procedures are given for solving this problem both graphically and computationally. It develops that these procedures also provide solutions for a much broader class of problems, containing other examples of practical interest.
ON THE SHORTEST SPANNING SUBTREE OF A GRAPH
AND THE TRAVELING SALESMAN PROBLEM

JOSEPH B. KRUSKAL, JR.

Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the length of the edge), and if these lengths are all distinct, then among the spanning trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the “matrix of lengths” of the graph, that is, the matrix \(|a_{ij}|\) where \(a_{ij}\) is the length of the edge connecting vertices \(i\) and \(j\). Of course, it is assumed that \(a_{ij}=a_{ji}\) and that \(a_{ii}=0\) for all \(i\) and \(j\).)

The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.
Minimum Spanning Tree Algos


A Randomized Linear-Time Algorithm to Find Minimum Spanning Trees

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Abstract. We present a randomized linear-time algorithm to find a minimum spanning tree in a connected graph with edge weights. The algorithm uses random sampling in combination with a recently discovered linear-time algorithm for verifying a minimum spanning tree. Our computational model is a unit-cost random-access machine with the restriction that the only operations allowed on edge weights are binary comparisons.

Categories and Subject Descriptors: F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—computations on discrete structures; G.2.2 [Discrete Mathematics]: Graph Theory—graph algorithms, network problems, trees; G.3 [Probability and Statistics]: probabilistic algorithms (including Monte Carlo); I.5.3 [Pattern Recognition]: Clustering

General Terms: Algorithms

Additional Key Words and Phrases: Matroid, minimum spanning tree, network, randomized algorithm
Prim-Jarnik Algorithm

- Traverse $G$ starting at any node
  - Maintain priority queue of nodes
  - set priority to weight of the edge that connects them to MST
- Un-added nodes start with priority $\infty$
- At each step
  - Connect the node with lowest cost
  - Update ("relax") neighbors as necessary
- Stop when all nodes added to MST
Example

Random node set to cost 0

\[ PQ = [(0, A), (∞, B), (∞, C), (∞, D), (∞, E), (∞, F)] \]
Example

Dequeue from PQ and update neighbors

\[ PQ = \{ (4, B), (5, D), (\infty, C), (\infty, E), (\infty, F) \} \]
Example

Dequeue from PQ and update neighbors

\[ PQ = [(4, C), (4, D), (6, E), (8, F)] \]
Example

\[ PQ = [(2, E), (4, D), (8, F)] \]
Example

\[ \text{PQ} = [(4, D), (4, F)] \]

Dequeue from PQ and update neighbors
Example

Dequeue from PQ and update neighbors

\[ PQ = [(3, F)] \]
Example

PQ = [ ]

Dequeue from PQ and update neighbors
Example
function `prim(G)`:  
// Input: weighted, undirected graph G with vertices V  
// Output: list of edges in MST  
for all v in V:  
    v.cost = ∞  
    v.prev = null  
  s = a random v in V  // pick a random source s  
  s = 0  
  MST = []  
  PQ = PriorityQueue(V)  // priorities will be v.cost values  
while PQ is not empty:  
    v = PQ.removeMin()  
    if v.prev != null:  
        MST.append((v, v.prev))  
    for all incident edges (v,u) of v such that u is in PQ:  
        if u.cost > (v,u).weight:  
            u.cost = (v,u).weight  
            u.prev = v  
            PQ.decreaseKey(u, u.cost)  
return MST
Simulate Prim-Jarnik

```java
function prim(G):
    // Input: weighted, undirected graph G with vertices V
    // Output: list of edges in MST
    for all v in V:
        v.cost = \infty
        v.prev = null
    s = a random v in V // pick a random source s
    s = 0
    MST = []
    PQ = PriorityQueue(V) // priorities will be v.cost values
    while PQ is not empty:
        v = PQ.removeMin()
        if v.prev != null: //guarantees we don’t add (s, s.prev)
            MST.append((v, v.prev))
        for all incident edges (v,u) of v such that u is in PQ:
            if u.cost > (v,u).weight:
                u.cost = (v,u).weight
                u.prev = v
                PQ.decreaseKey(u, u.cost)
    return MST
```
Simulate Prim-Jarnik

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Simulate Prim-Jarnik

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  // Output: list of edges in MST
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    v.cost = ∞
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  s = 0
  MST = []
  PQ = PriorityQueue(V) // priorities will be v.cost values
  while PQ is not empty:
    v = PQ.removeMin()
    if v.prev != null: //guarantees we don’t add (s, s.prev)
      MST.append((v, v.prev))
    for all incident edges (v,u) of v such that u is in PQ:
      if u.cost > (v,u).weight:
        u.cost = (v,u).weight
        u.prev = v
        PQ.decreaseKey(u, u.cost)
  return MST
Simulate Prim-Jarnik

function $\text{prim}(G)$:
  // Input: weighted, undirected graph $G$ with vertices $V$
  // Output: list of edges in MST
  for all $v$ in $V$:
    $v$.cost = $\infty$
    $v$.prev = null
  $s$ = a random $v$ in $V$ // pick a random source $s$
  $s$ = 0
  MST = []
  PQ = PriorityQueue($V$) // priorities will be $v$.cost values
  while PQ is not empty:
    $v$ = PQ.removeMin()
    if $v$.prev != null: //guarantees we don’t add $(s, s$.prev)$
      MST.append($v$, $v$.prev)
      for all incident edges $(v,u)$ of $v$ such that $u$ is in PQ:
        if $u$.cost > $(v,u)$.weight:
          $u$.cost = $(v,u)$.weight
          $u$.prev = $v$
          PQ.decreaseKey($u$, $u$.cost)
  return MST
Runtime of Prim-Jarnik

Activity #2

2 min
Runtime of Prim-Jarnik

Activity #2

2 min
Runtime of Prim-Jarnik

Activity #2
Runtime of Prim-Jarnik

0 min

Activity #2
Runtime Analysis

- Decorating nodes with distance and previous pointers is $O(|V|)$
- Putting nodes in PQ is $O(|V| \log |V|)$ (really $O(|V|)$ since $\infty$ priorities)
- While loop runs $|V|$ times
  - removing vertex from PQ is $O(\log |V|)$
  - So $O(|V| \log |V|)$
- For loop (in while loop) runs $|E|$ times in total
  - Replacing vertex’s key in the PQ is $\log |V|$
  - So $O(|E| \log |V|)$
- Overall runtime
  - $O(|V| + |V| \log |V| + |V| \log |V| + |E| \log |V|)$
  - $= O((|E| + |V|) \log |V|)$
Proof of Correctness

- Common way of proving correctness of greedy algs
  - show that algorithm is always correct at every step
- Best way to do this is by induction
  - tricky part is coming up with the right invariant
Graph Cuts

- A cut is any partition of the vertices into two groups

Here $G$ is partitioned in 2

- with edges $b$ and $a$ joining the partitions
Proof of Correctness

- **P(n)**
  - first \( n \) edges added by Prim are a subtree of some MST
- Base case when \( n=0 \)
  - no edges have been added yet so \( P(0) \) is trivially true
- Inductive Hypothesis
  - first \( k \) edges added by Prim form a tree \( T \) which is subtree of some MST \( M \)
Proof of Correctness

- Inductive Step
  - Let $e$ be the $(k+1)$th edge that is added
  - $e$ will connect $T$ (green nodes) to an unvisited node (one of blue nodes)
  - We need to show that adding $e$ to $T$
    - forms a subtree of some MST $M'$
    - (which may or may not be the same MST as $M$)
Proof of Correctness

- Two cases
  - $e$ is in original MST $M$
  - $e$ is not in $M$
- Case 1: $e$ is in $M$
  - there exists an MST that contains first $k+1$ edges
  - So $P(k+1)$ is true!
Proof of Correctness

- Case 2: $e$ is not in $M$
  - if we add $e = (u, v)$ to $M$ then we get a cycle
  - why? since $M$ is span. tree there must be path from $u$ to $v$ w/o $e$
  - so there must be another edge $e'$ that connects $T$ to unvisited nodes
  - We know $e.weight \leq e'.weight$ because Prim chose $e$ first
Proof of Correctness

- So if we add $e$ to $M$ and remove $e'$
  - we get a new MST $M'$ that is no larger than $M$ and contains $T$ & $e$

- $P(k+1)$ is true
  - because $M'$ is an MST that contains the first $k+1$ edges added by Prim's
Proof of Correctness

- Since we have shown
  - $P(0)$ is true
  - $P(k+1)$ is true assuming $P(k)$ is true (for both cases)
- The first $n$ edges added by Prim form a subtree of some MST
Outline

- Minimum Spanning Trees
  - Analysis
- Prim-Jarnik Algorithm
  - Analysis
  - Proof of Correctness
- Kruskal’s Algorithm
  - Union-Find
  - Analysis
Kruskal’s Algorithm

- Sort edges by weight in ascending order
- For each edge in sorted list
  - If adding edge does not create cycle…
  - …add it to MST
- Stop when you have gone through all edges
Example

edges = [(C,E), (D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
Simulate Kruskal

Activity #3

2 min
Simulate Kruskal

Activity #3

2 min
Simulate Kruskal

1 min

Activity #3
Simulate Kruskal

0 min

Activity #3
Kruskal

- How can we tell if adding edge will create cycle?
- Start by giving each vertex its own “cloud”
- If both ends of lowest-cost edge are in same cloud
  - we know that adding the edge will create a cycle!
- When edge is added to MST
  - merge clouds of the endpoints
Example

edges = [(C,E), (D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
Example

edges = [(D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
edges = [(B,C),(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
Example

edges = [(E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
Example

edges = [(B, D), (A, B), (A, D), (B, E), (B, F)]
Example

edges = [(A,B), (A,D), (B,E), (B,F)]

BD cannot be added because it would lead to a cycle
Example

edges = [(A,D), (B,E), (B,F)]
Example

edges = [(B, E), (B, F)]

AD cannot be added because it would lead to a cycle
Example

BE cannot be added because it would lead to a cycle

edges = [(B,F)]
BF cannot be added because it would lead to a cycle

edges = [ ]
Kruskal Pseudo-Code

function kruskal(G):
    // Input: undirected, weighted graph G
    // Output: list of edges in MST
    for vertices v in G:
        makeCloud(v) // put every vertex into it own set
    MST = []
    Sort all edges
    for all edges (u,v) in G sorted by weight:
        if u and v are not in same cloud:
            add (u,v) to MST
            merge clouds containing u and v
    return MST
Merging Clouds (Naive way)

- Assign each vertex a different number
  - that represents its initial cloud
- To merge clouds of \( u \) and \( v \)
  - Find all vertices in each cloud
  - Figure out which of the clouds is smaller
  - Redecorate all vertices in smaller cloud w/ bigger cloud’s number
Merging Clouds (Naive way)

- Finding all vertices in \( u \) & \( v \)'s clouds is \( O( |V| ) \)
  - because we have to iterate through each vertex...
  - …and check if its cloud number matches \( u \) or \( v \)'s cloud number

- Figuring out smaller cloud is \( O(1) \)
  - as long as we keep track of cloud size as we find vertices in them

- Changing cloud numbers of nodes in smaller cloud is \( O( |V| ) \)
  - because smallest cloud could be as big as \( |V| / 2 \) vertices

- Total runtime to merge clouds
  - \( O( |V| + 1 + |V| ) = O( |V| ) \)
Runtime of Naive Kruskal

- Finding all vertices in \( u \) & \( v \)'s clouds is \( O( |V|) \)
  - because we have to iterate through each vertex...
  - ...and check if its cloud number matches \( u \) or \( v \)'s cloud number

- Figuring out smaller cloud is \( O(1) \)
  - as long as we keep track of cloud size as we find vertices in them

- Changing cloud numbers of vertices in smaller cloud is \( O( |V|) \)
  - because cloud could be as big as \( |V|/2 \) vertices

- Merge Runtime
  - \( O( |V|) + O(1) + O( |V|) = O( |V|) \)
Runtime of Naive Kruskal

- Finding all vertices in $u$ & $v$'s clouds is $O(\mid V \mid)$
  - because we have to iterate through each vertex...
  - ...and check if its cloud number matches $u$ or $v$'s cloud number
- Figuring out smaller cloud is $O(1)$
  - as long as we keep track of cloud size as we find vertices in them
- Changing cloud numbers of vertices in smaller cloud is $O(\mid V \mid)$
  - because cloud could be as big as $\mid V \mid / 2$ vertices
- Merge Runtime
  - $O(\mid V \mid) + O(1) + O(\mid V \mid) = O(\mid V \mid)$

Activity #4

2 min
Runtime of Naive Kruskal

- Finding all vertices in \( u \) & \( v \)'s clouds is \( O(|V|) \)
  - because we have to iterate through each vertex…
  - …and check if its cloud number matches \( u \) or \( v \)'s cloud number
- Figuring out smaller cloud is \( O(1) \)
  - as long as we keep track of cloud size as we find vertices in them
- Changing cloud numbers of vertices in smaller cloud is \( O(|V|) \)
  - because cloud could be as big as \( |V|/2 \) vertices
- Merge Runtime
  - \( O(|V|) + O(1) + O(|V|) = O(|V|) \)
Runtime of Naive Kruskal

- Finding all vertices in u & v's clouds is $O(|V|)$
  - because we have to iterate through each vertex…
  - …and check if its cloud number matches u or v’s cloud number
- Figuring out smaller cloud is $O(1)$
  - as long as we keep track of cloud size as we find vertices in them
- Changing cloud numbers of vertices in smaller cloud is $O(|V|)$
  - because cloud could be as big as $|V|/2$ vertices
- Merge Runtime
  - $O(|V|) + O(1) + O(|V|) = O(|V|)$

Activity #4

0 min
function \texttt{kruskal}(G):

// Input: undirected, weighted graph G
// Output: list of edges in MST

for vertices \( v \) in \( G \):
    makeCloud(\( v \))

\( \text{MST} = [\ ] \)

Sort all edges

for all edges \((u,v)\) in \( G \) sorted by weight:
    if \( u \) and \( v \) are not in same cloud:
        add \((u,v)\) to \( \text{MST} \)
        merge clouds containing \( u \) and \( v \)

return \( \text{MST} \)
Kruskal Runtime

- $O(|V|)$ for iterating through vertices
- $O(|E| \log |E|)$ for sorting edges
- $O(|E| \times |V|)$ for iterating through edges and merging clouds naively
- $O(|V| + |E| \log |E| + |E| \times |V|)$
  - $= O(|E| \times |V|) = O(|V|^2 \times |V|) = O(|V|^3)$
- Can we do better?

since $|E| \leq |V|^2$
Union-Find

- Let's rethink notion of clouds
  - instead of labeling vertices w/ cloud numbers
  - think of clouds as small trees
- Every vertex in these trees has
  - a parent pointer that leads up to root of the tree
  - a rank that measures how deep the tree is
Example

edges = [(C,E), (D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
edges = [(D,F), (B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
Example

edges = [(B,C), (E,F), (B,D), (A,B), (A,D), (B,E), (B,F)]
edges = [(E,F),(B,D),(A,B),(A,D),(B,E),(B,F)]
Example

edges = [(B,D), (A,B), (A,D), (B,E), (B,F)]
Example

edges = [(A,D), (B,E), (B,F)]
Example

edges = [(A,D), (B,E), (B,F)]
Implementing Union-Find

- At start of Kruskal
  - every node is put into own cloud

```cpp
// Decorates every vertex with its parent ptr & rank
function makeCloud(x):
    x.parent = x
    x.rank = 0
```

0 [A] 0 [B]
Implementing Union-Find

- Suppose A is in cloud 1 and B is in cloud 2
- Instead of relabeling B as cloud 1 make B point to A
  - Think of this as the union of two clouds

  ![Diagram of union of two clouds]

- Given two clouds which one should point to the other?
Implementing Union-Find

- We use the rank to decide
  - make lower-ranked root point to higher-ranked root
  - then update rank
- How do we update ranks?
  - For clouds of size 1 root always has rank 0
  - For clouds of size larger than 1 we increment rank only when merging clouds of same rank
Implementing Union-Find

- Merging trees with same rank
Implementing Union-Find

- Merging trees with same rank
Implementing Union-Find

- Merging trees with different ranks
Implementing Union-Find

- Merging trees with different ranks
// Merges two clouds, given the root of each cloud
function union(root1, root2):
    if root1.rank > root2.rank:
        root2.parent = root1
    elif root1.rank < root2.rank:
        root1.parent = root2
    else:
        root2.parent = root1
        root1.rank++
Implementing Union-Find

- To find the cloud of B
  - follow B’s parent pointer all the way up to root

```cpp
// Finds the cloud of a given vertex
function find_root(x):
    while x.parent != x:
        x = x.parent
    return x
```
Path Compression

- This approach to implementing `find` runs in $O(\log|V|)$
- Not obvious to see why and proof beyond CS16
- We can bring this down to amortized $O(1)$
  - With path compression...
  - ...a way of flattening the structure of the tree...
  - ...whenever `find()` is used on it
Path Compression

- Instead of traversing up tree every time D's cloud is asked for
  - We only search for D's root once
  - As we follow chain of parents to A we set parents of D & C to A

\[
\text{Amortized } O(1)
\]

\[
O(\log |V|)
\]
function `find_root(x)`:
   if x.parent != x:
      x.parent = find_root(x.parent)
   return x.parent
Runtime of Kruskal w/ Path Compression

Activity #5

1 min
Runtime of Kruskal w/ Path Compression

Activity #5

1 min
Runtime of Kruskal w/ Path Compression

Activity #5

0 min
function **kruskal**\( (G) \):

// Input: undirected, weighted graph G
// Output: list of edges in MST

for vertices \( v \) in \( G \):
    makeCloud\( (v) \)

\( \text{MST} = [\] \)

Sort all edges

for all edges \( (u,v) \) in \( G \) sorted by weight:
    if \( u \) and \( v \) are not in same cloud:
        add \( (u,v) \) to \( \text{MST} \)
        merge clouds containing \( u \) and \( v \)

return \( \text{MST} \)

\( O(|V|) \)

\( O(|E|) \)

\( O(|E| \log |E|) \)

**amortized**
Kruskal Runtime

- \( O(|V|) \) for iterating through vertices
- \( O(|E| \log |E|) \) for sorting edges
- \( O(|E| \times 1) \) for iterating through edges and merging clouds with path compression

\[ O(|V| + |E| \log |E| + |E| \times 1) \]

\[ = O(|V| + |E| \log |E|) \]

- \( O(|V| + |E| \log |E|) \) better than \( O(|V|^3) \)
Readings

- Dasgupta Section 5.1
  - Explanations of MSTs
  - and both algorithms discussed in this lecture