Directed Acyclic Graphs & Topological Sort

CS16: Introduction to Data Structures & Algorithms

Spring 2019
Outline

- Directed Acyclic Graphs
- Topological Sort
  - Hand-simulation
  - Pseudo-code
  - Runtime analysis
Directed Acyclic Graphs

- A DAG is **directed** & **acyclic**
- Directed
  - edges have origin & destination…
  - ….represented by a directed arrow
- Acyclic
  - No cycles!
  - Starting from any vertex, there is no path that leads back to the same vertex
Directed Acyclic Graphs

- DAGs often used to model situations in which completing certain things depend on completing other things
  - ex: course prerequisites or small tasks in a big project

- Terminology
  - Sources: vertices with no incoming edges (no dependencies)
  - Sinks: vertices with no outgoing edges
  - In-degree of a vertex: number of incoming edges of the vertex
  - Out-degree of a vertex: number of outgoing edges of the vertex
Directed Acyclic Graphs — Example
Imagine you are a CS concentrator
You need to plan your courses for next 3 years
How can you do that taking into account pre-requisites?
- Represent courses w/ a DAG
- Use topological sort!
  - Produces topological ordering of a DAG
Topological Sort

- Topological Ordering
  - ordering of vertices in DAG...
  - ...such that for each vertex v...
  - ...all of v's prereqs come before it in the ordering

- Topological Sort
  - Algorithm that produces topological ordering given a DAG

- Valid topological orderings
  - 15, 16, 22, 141
  - 22, 15, 16, 141
  - 15, 22, 16, 141
Topological Sort—General Strategy

- If vertex has no prerequisites (i.e., is a source), we can visit it!
- Once we visit a vertex,
  - all of its outgoing edges can be deleted
  - because that prerequisite has been satisfied
- Deleting edges might create new sources
  - which we can now visit
- Data Structures needed
  - DAG to top-sort
  - A structure to keep track of sources
  - A list to keep track of the resultant topological ordering
Topological Sort—Simulation

List:

15 16 22 33 141 123 224 Stack
Topological Sort—Simulation

Populate Stack with source vertices

List:
Topological Sort—Simulation

Pop from stack and add to list

List: 15
Topological Sort—Simulation

Remove outgoing edges & check corresponding vertices

List: 15

Stack
Topological Sort—Simulation

16 has no more incoming edges so push it on the stack

List: 15

Stack

16

22
Topological Sort—Simulation

Pop from the stack and add to list

List: 15 16

Stack:

22
141
33
123
224
Topological Sort—Simulation

Remove outgoing edges & check the corresponding vertices

List:  15  16

Stack
Topological Sort—Simulation

33 has no more incoming edges so push it onto the stack.
141 still has an incoming edge.

# of incoming edges = 1
Topological Sort—Simulation

Pop from the stack & repeat!
Topological Sort—Simulation

List: 15 16 33

Stack:

123 → 224

15

16

33

22
Topological Sort—Simulation

List: 15 16 33

Stack: 123 224

123 22
Topological Sort—Simulation

List: 15 16 33 123

Stack: 22

Numbers: 22 141 15 16 33 123 224
Topological Sort—Simulation

List: 15 16 33 123

Stack: 22 141 123 33 15 16 224 224
Topological Sort—Simulation

List: 15 16 33 123 224

Stack: 22
Topological Sort—Simulation

List: 15  16  33  123  224  22

Stack
Topological Sort—Simulation
Topological Sort—Simulation

We're done!

List: 15 16 33 123 224 22 141

Stack
Topological Sort—Stack

Activity #1

2 min
Topological Sort—Stack

Activity #1

2 min
Topological Sort—Stack

Activity #1
Topological Sort—Stack

Activity #1
function **top_sort**(graph g):
  // Input: A DAG g
  // Output: A list of vertices of g, in topological order
  s = Stack()
  l = List()
  for each vertex in g:
    if vertex is source:
      s.push(vertex)
  while s is not empty:
    v = s.pop()
    l.append(v)
    for each outgoing edge e from v:
      w = e.destination
      delete e
      if w is a source:
        s.push(w)
  return l
Topological Sort Runtime

function top_sort(graph g):
    // Input: A DAG g
    // Output: A list of vertices of g, in topological order
    s = Stack()
    l = List()
    for each vertex in g:
        if vertex is source:
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    while s is not empty:
        v = s.pop()
        l.append(v)
        for each outgoing edge e from v:
            w = e.destination
            delete e
            if w is a source:
                s.push(w)
    return l

Looping through every vertex to find sources is $O(|V|)$
Topological Sort Runtime

function *top_sort*(graph \( g \)):

// Input: A DAG \( g \)
// Output: A list of vertices of \( g \), in topological order

\( s = \text{Stack}() \)
\( l = \text{List}() \)

for each vertex in \( g \):
    if vertex is source:
        \( s.push(\text{vertex}) \)

while \( s \) is not empty:
    \( v = s.pop() \)
    \( l.append(v) \)

for each outgoing edge \( e \) from \( v \):
    \( w = e.\text{destination} \)
    delete \( e \)
    if \( w \) is a source:
        \( s.push(\text{w}) \)

return \( l \)

Looping through every vertex to find sources is \( O(|V|) \)

Stack will hold each vertex once

At each iteration we only visit outgoing edges from popped vertex. So every edge visited once.

Total runtime: \( O(|V| + |E|) \)
Topological Sort—Queue

Activity #2
Topological Sort—Queue

Activity #2

2 min
Topological Sort—Queue
Topological Sort—Queue

Activity #2
Topological Sort Variations

- What if we're not allowed to modify original DAG?
  - How do we delete edges?
  - Use decorations!
- Start by decorating each vertex with it's in-degree
  - Instead of deleting edge
  - decrement in-degree of destination vertex by 1
  - then push vertex on stack when in-degree is 0!
Topological Sort Variations

- Do we need to use a stack?
  - No! Any data structure like a list or queue would work
  - All we're doing is keeping track of sources
- Different structures might yield different topological orderings
  - Why do they all work?
  - Vertices are only added to structure when they become a source
    - i.e., when all of it’s "prerequisites" have been visited
  - This invariant is maintained throughout algorithm…
  - …and guarantees a valid topological ordering!
Topological Sort

Activity #3

2 min
Topological Sort

2 min

Activity #3
Topological Sort
Topological Sort

Activity #3
Top Sort: Why only on DAGs?

- If the graph has a cycle...

...we don't have a valid topological ordering

- We can use top sort to check if a DAG has a cycle
- Run top sort on graph
  - if there are edges left at the end but no more sources
  - then there must be a cycle