Graphs

CS16: Introduction to Data Structures & Algorithms
Spring 2020

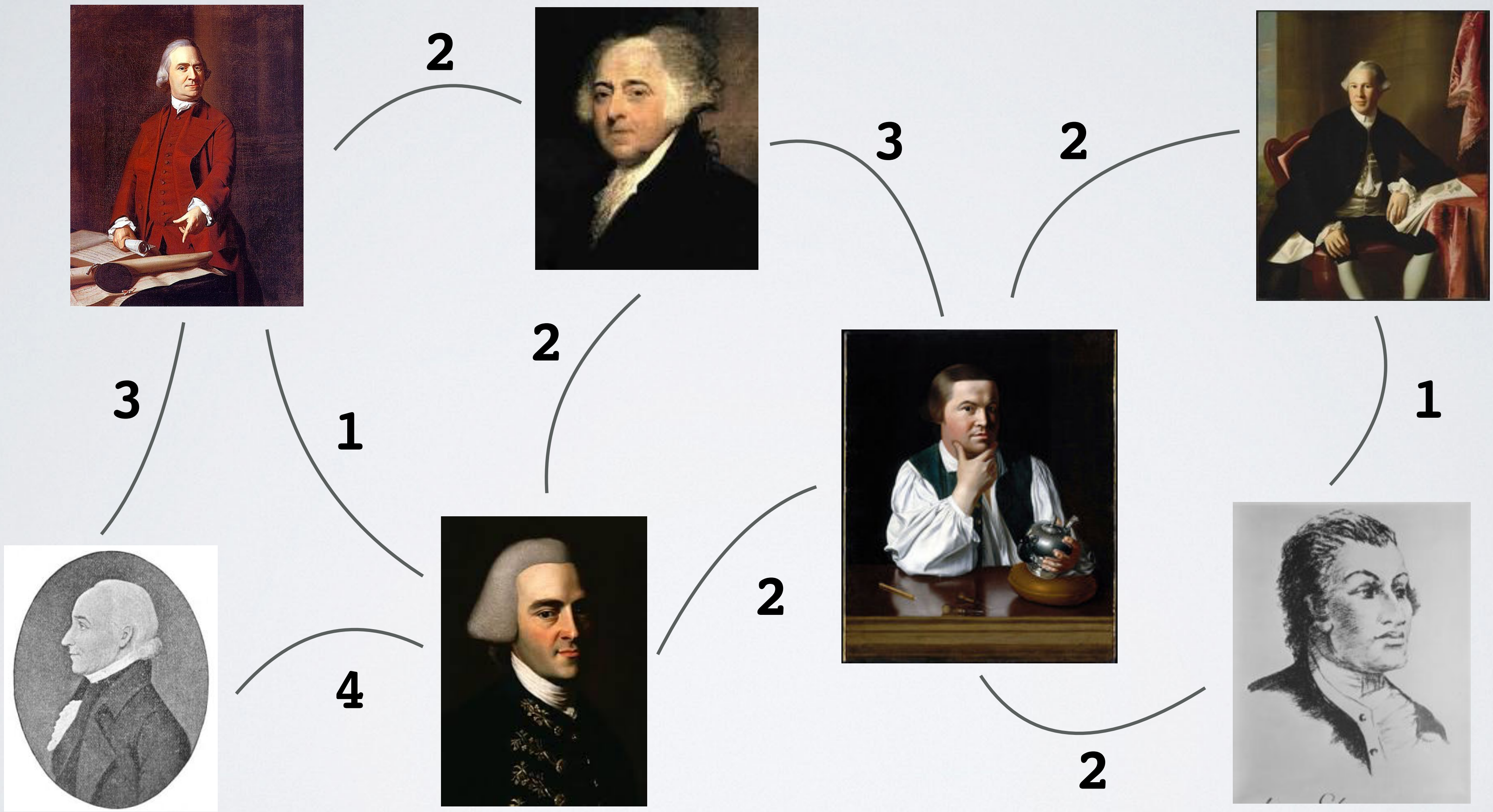
Outline

- What is a Graph
- Terminology
- Properties
- Graph Types
- Representations
- Performance
- BFS/DFS
- Applications

What is a Graph

- A graph is defined by
  - a set of vertices (or vertexes, or nodes) **`V`**
  - a set of edges **`E`**
- Vertices and edges can both store data

Example: Social Graph

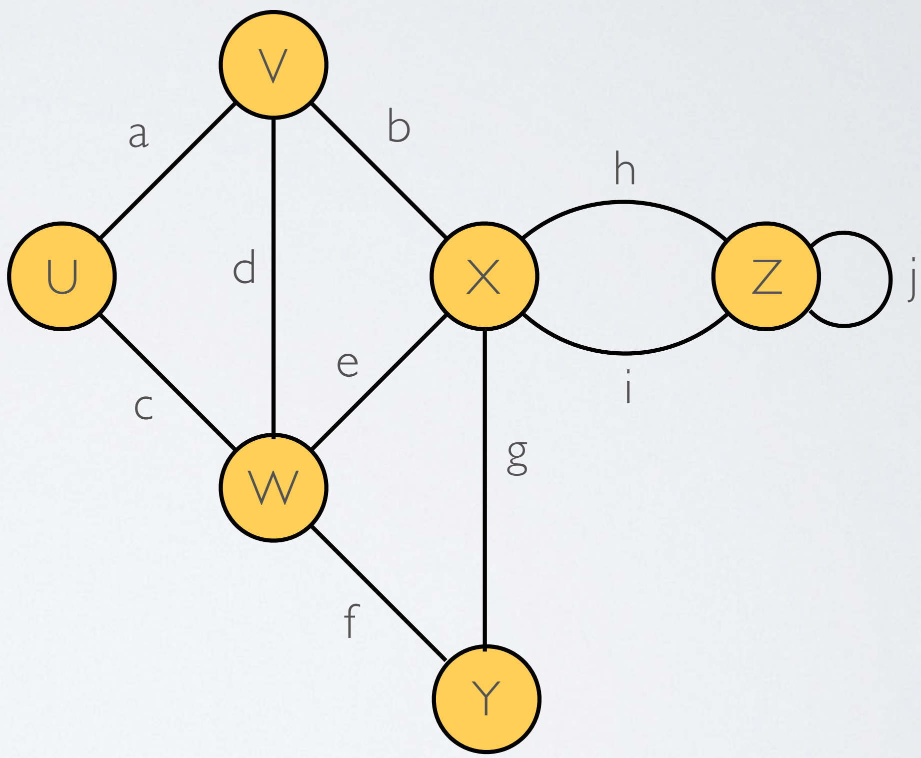


Kieran Healy, "Using metadata to find Paul Revere"

https://kieranhealy.org/blog/archives/2013/06/09/using-metadata-to-find-paul-revere/

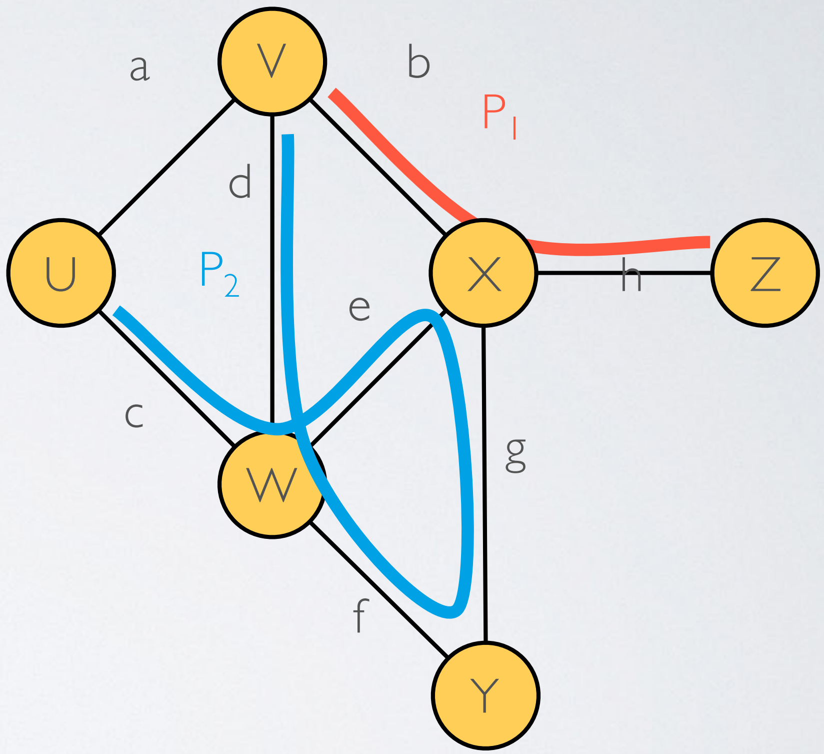
Terminology

- Endpoints or end vertices of an edge
  - **U** and **V** are endpoints of edge **a**
- Incident edges of a vertex
  - **a, b, d** are incident to **V**
- Adjacent vertices
  - **U** and **V** are adjacent
- Degree of a vertex
  - **X** has degree of 5
- Parallel (multiple) edges
  - **h, i** are parallel edges
- Self-loops
  - **j** is a self-looped edge



Terminology

- A path is a sequence of alternating vertices and edges
  - begins and ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are visited at most once
- Examples
  - $P_1 = V \rightarrow_b X \rightarrow_h Z$ is a simple path
  - $P_2 = U \rightarrow_c W \rightarrow_e X \rightarrow_g Y \rightarrow_f W \rightarrow_d V$ is not a simple path, but is still a path



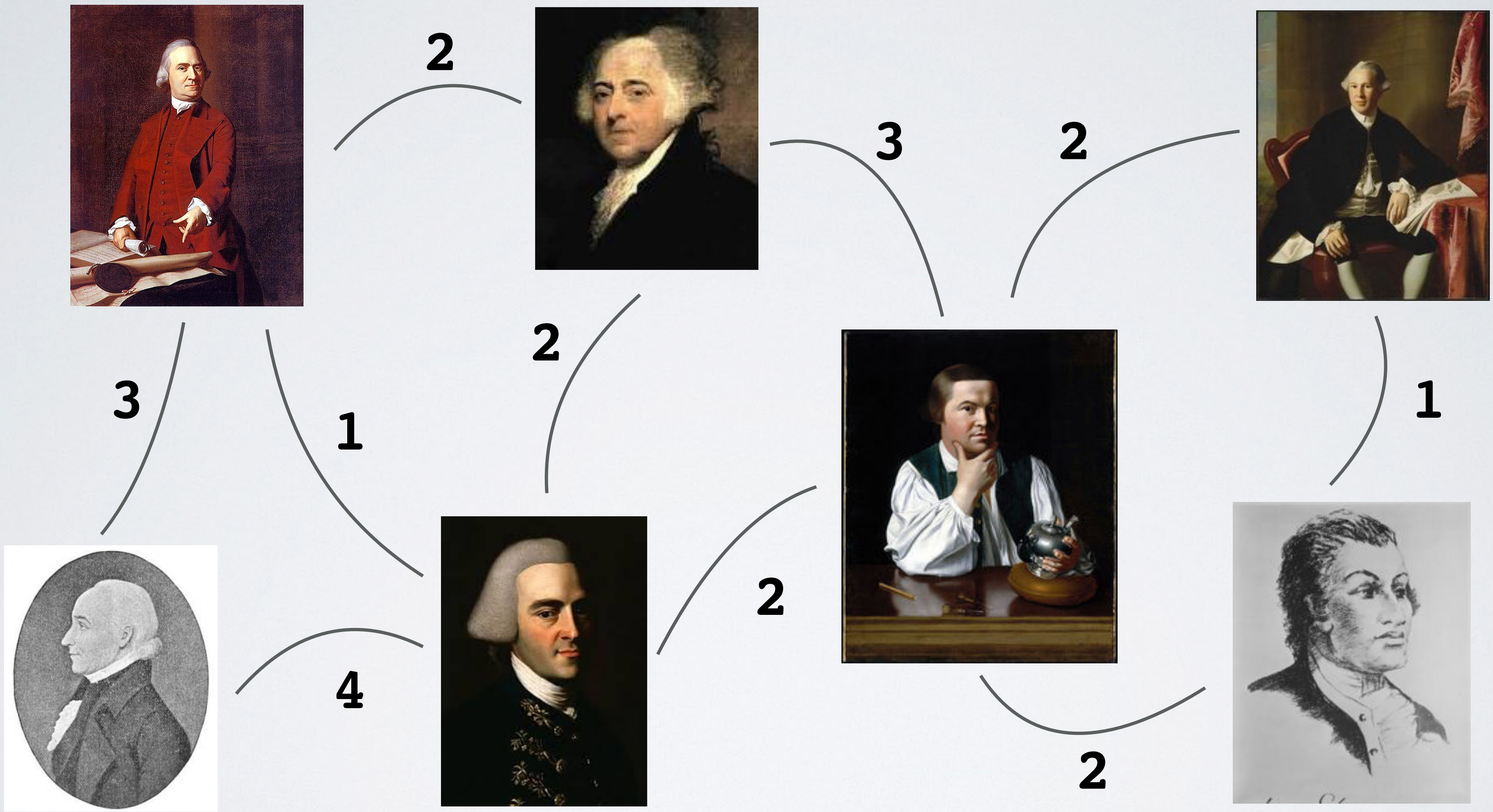
Applications

- Flight networks
- Road networks & GPS
- The Web
  - pages are vertices
  - links are edges
- The Internet
  - routers and devices are vertices
  - network connections are edges
- Facebook
  - profiles are vertices
  - friendships are edges

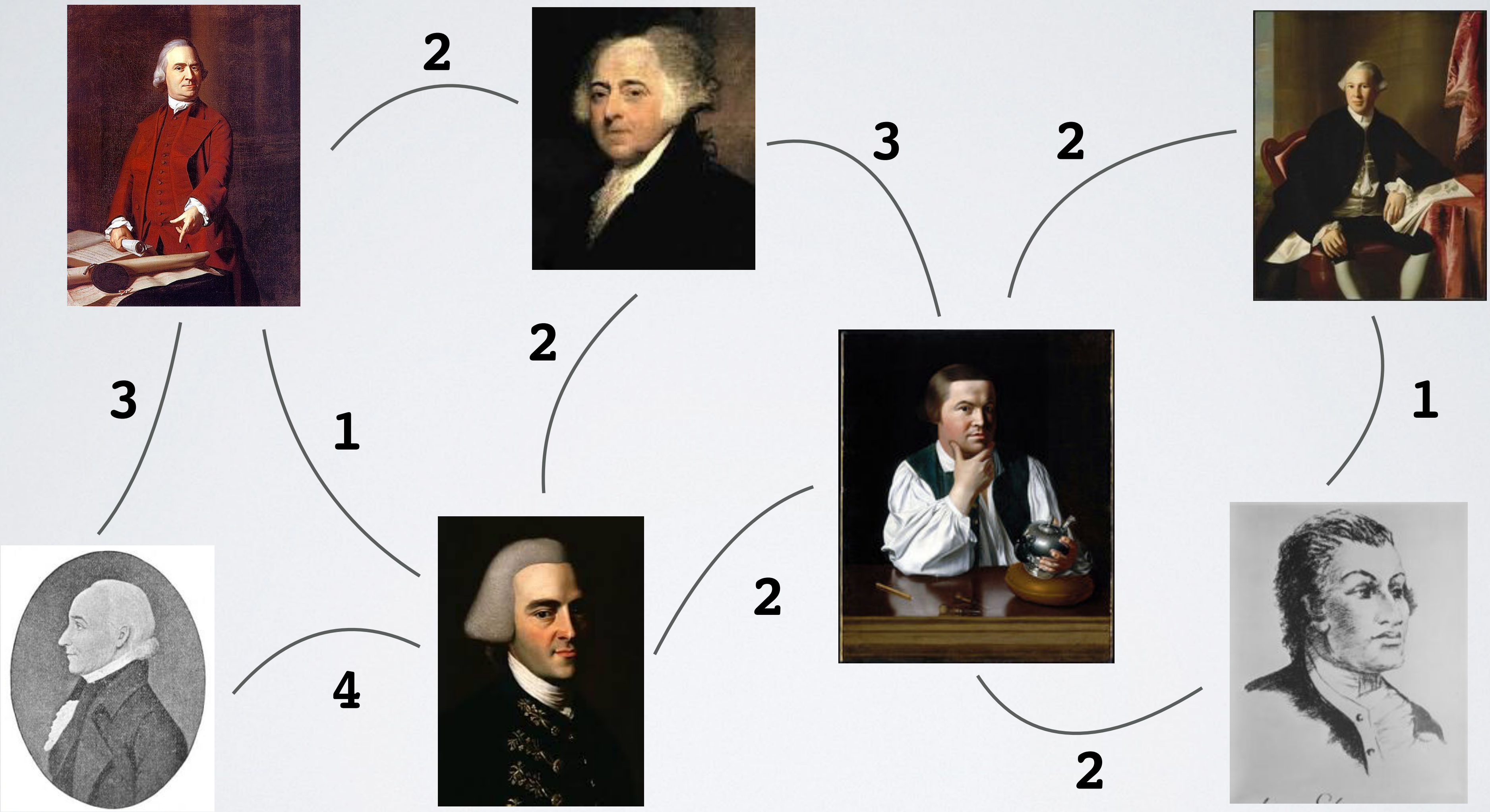
Graph Properties

- A graph `G'=(V',E')` is a **subgraph** of `G=(V,E)`
  - if `V'` ⊆ `V` and `E'` ⊆ `E`
- A graph is **connected** if
  - there exists path from each vertex to every other vertex
- A path is a **cycle** if
  - it starts and ends at the same vertex
- A graph is **acyclic**
  - if it has no cycles

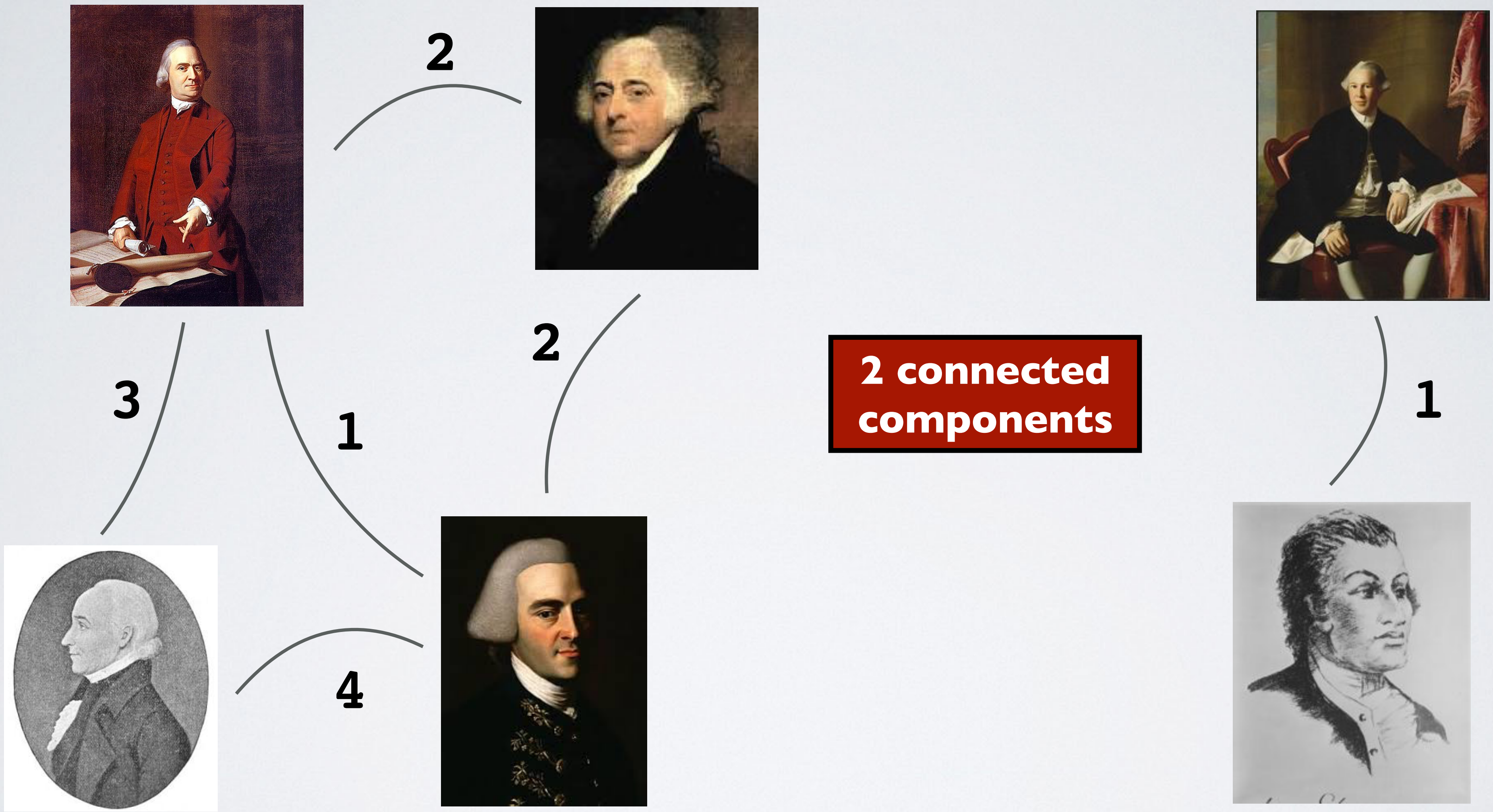
A Subgraph



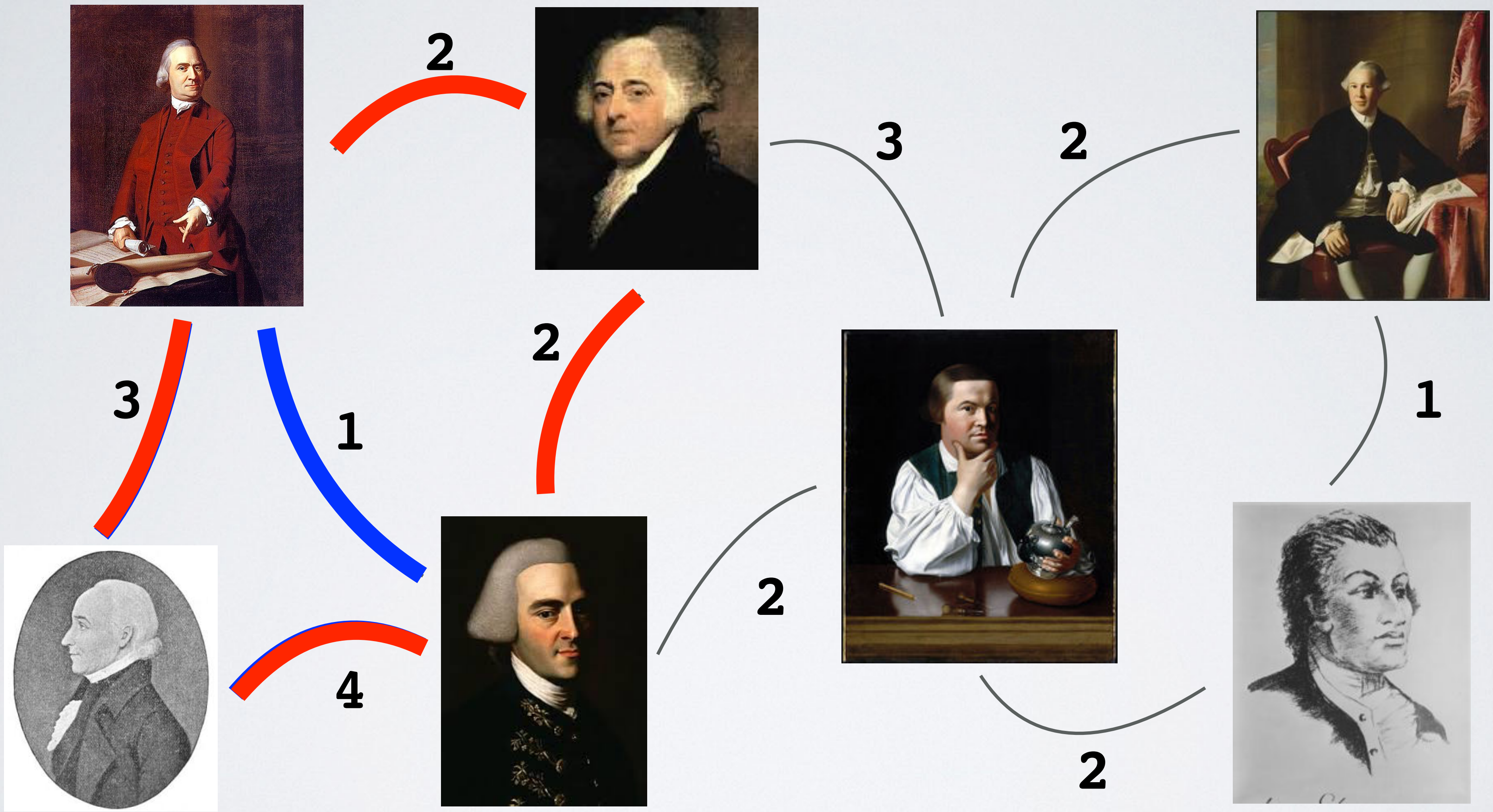
Connected?



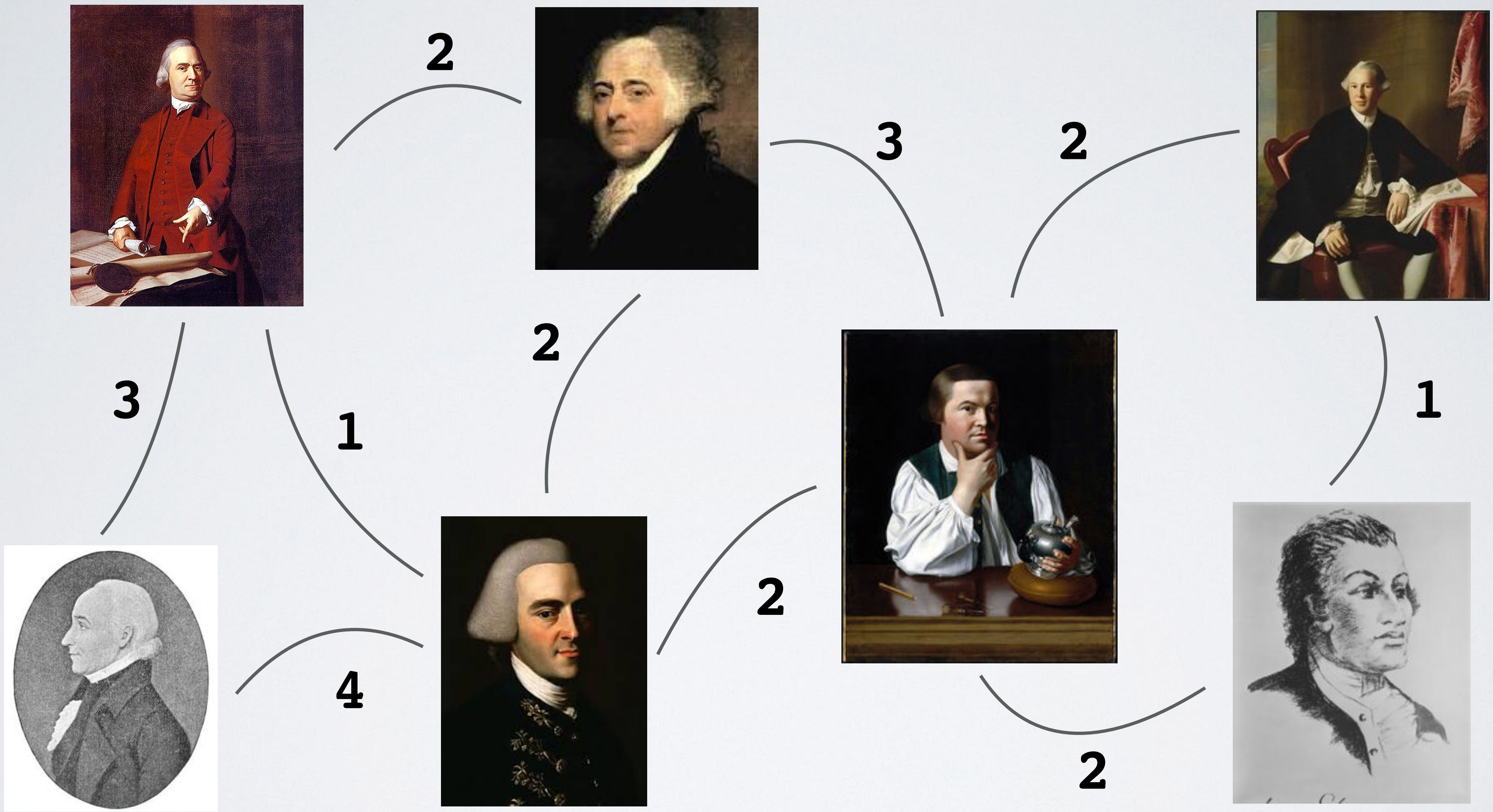
Connected?



Cycles



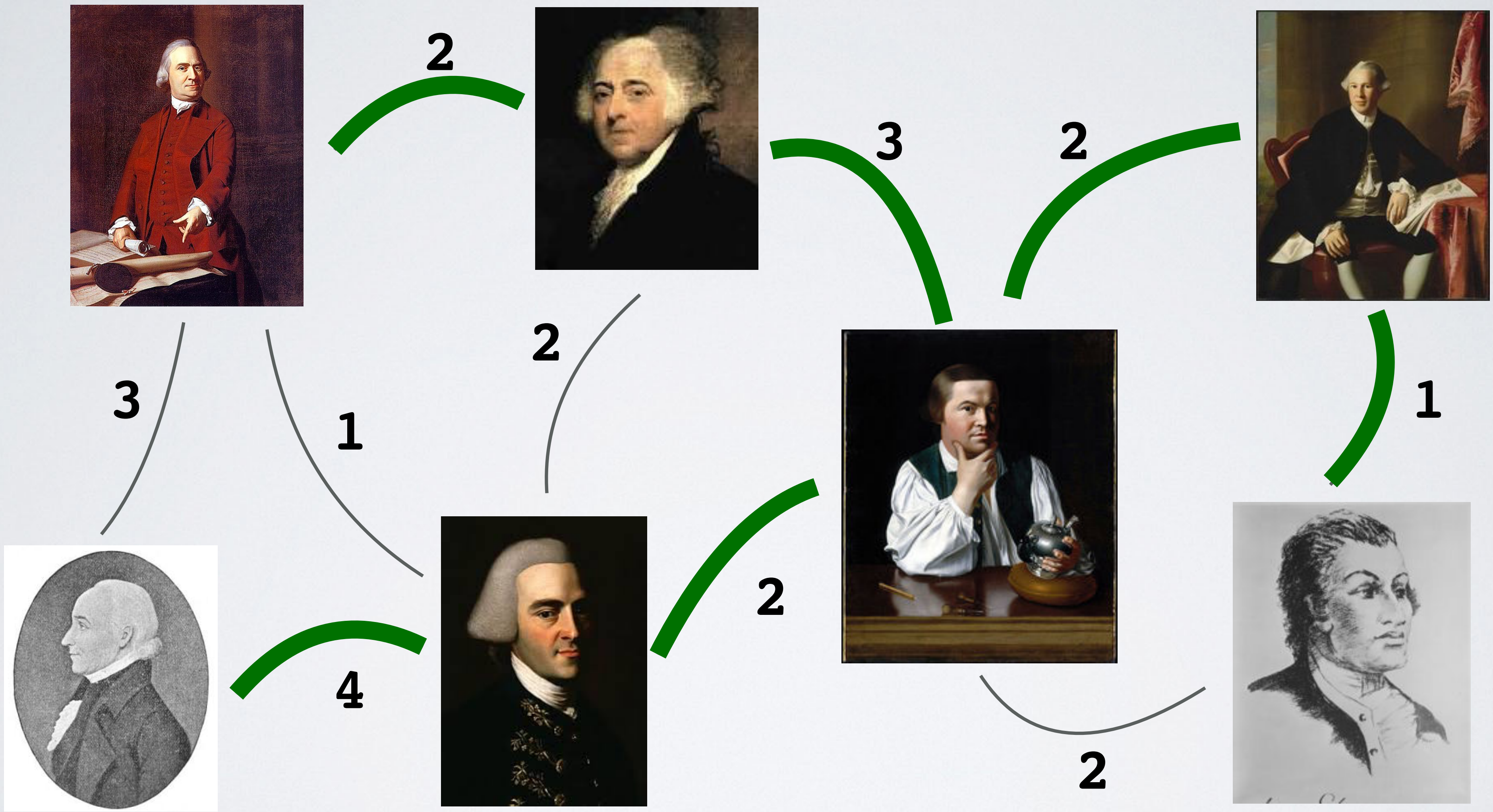
Acyclic?



Graph Properties

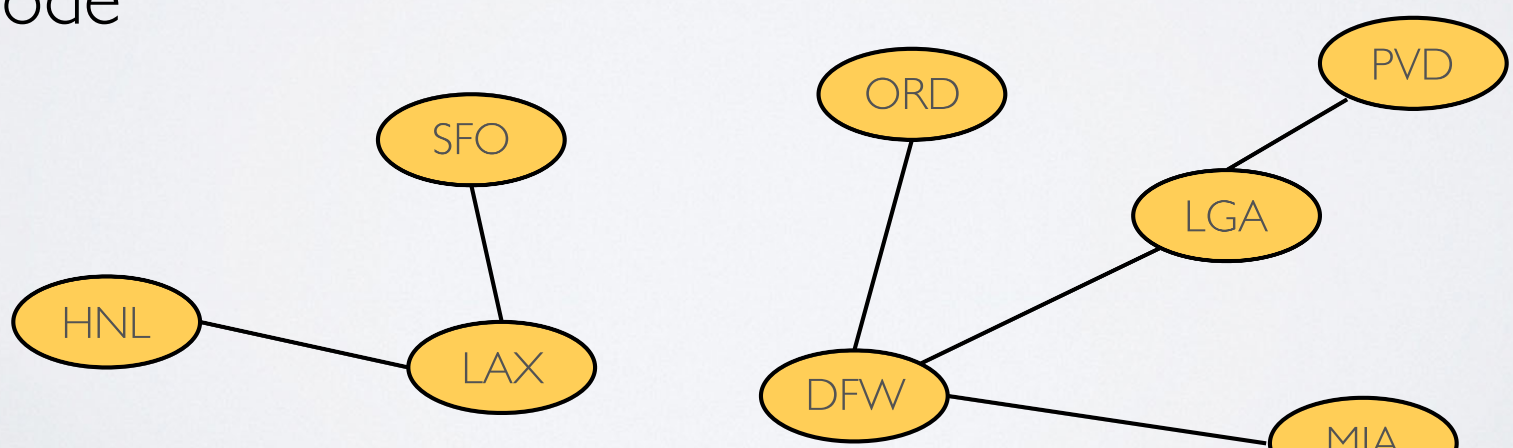
- A **spanning tree** of **G** is a subgraph with
  - all of **G**' s vertices in a single *tree*
  - and enough edges to connect each vertex w/o cycles

Spanning tree

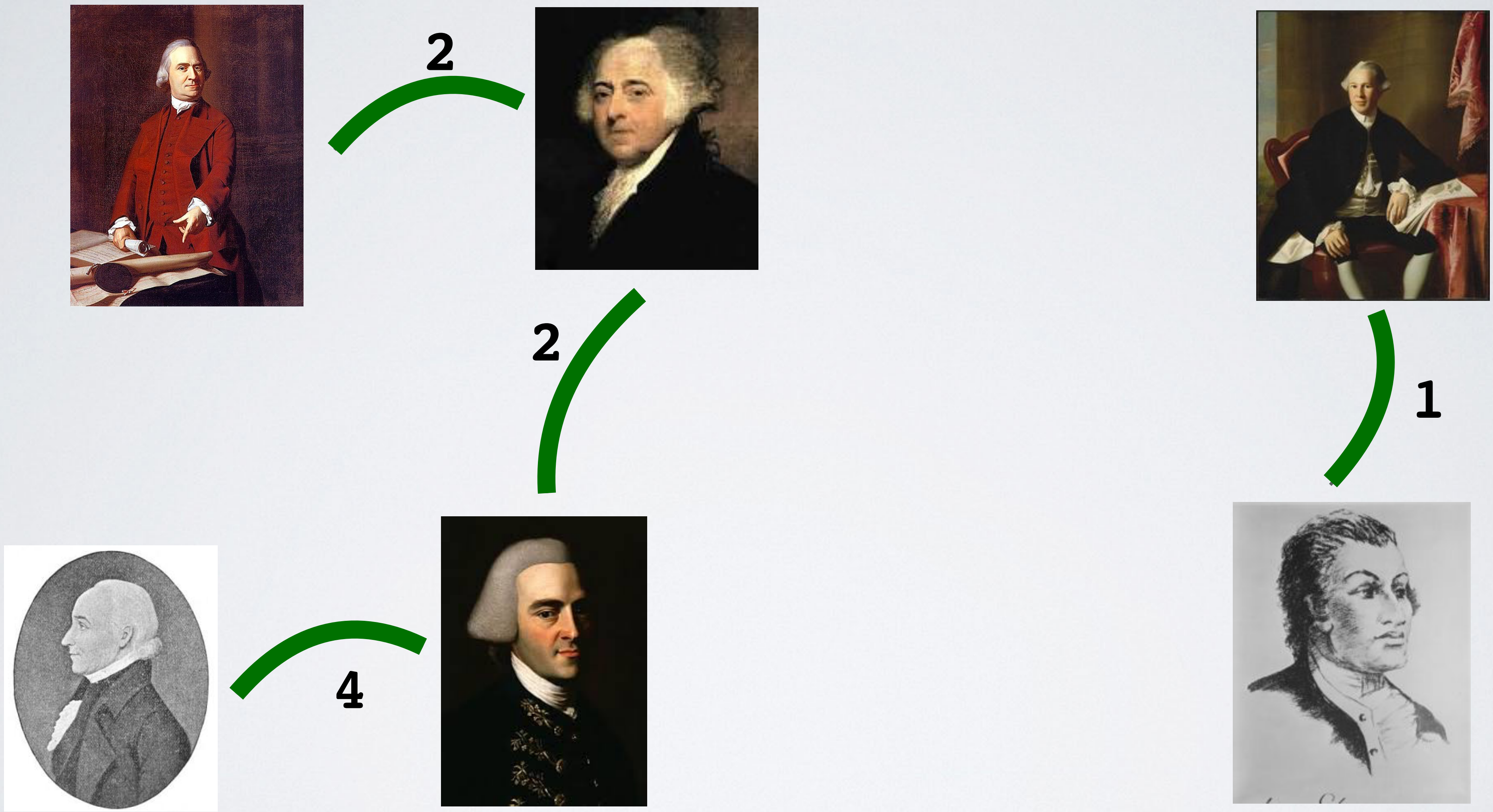


Graph Properties

- A **spanning forest** is
  - a subgraph that consists of a spanning tree in each connected component of graph
- Spanning forests never contain cycles
  - this might not be the "best" or shortest path to each node



Spanning forest



Graph Properties

- **G** is a tree if and only if it satisfies any of these conditions
  - **G** has `|V|-1` edges and no cycles
  - **G** has `|V|-1` edges and is connected
  - **G** is connected, but removing any edge disconnects it
  - **G** is acyclic, but adding any edges creates a cycle
  - Exactly one simple path connects each pair of vertices in **G**

Graph Proof I

- Prove that
  - the sum of the degrees of all vertices of some graph **G**…
  - …is twice the number of edges of **G**
- Let $V = \{v_1, v_2, \ldots, v_p\}$, where p is number of vertices
- The total sum of degrees D is such that
  - $D = deg(v_1) + deg(v_2) + \ldots + deg(v_p)$
- But each edge is counted twice in D
  - one for each of the two vertices incident to the edge
- So $D = 2|E|$, where $|E|$ is the number of edges.

Graph Proof 2

- Prove using induction that if G is connected then
  - $|E| \geq |V|-1$, for all $|V| \geq 1$
- Base case $|V|=1$
  - If graph has one vertex then it will have 0 edges
  - so since $|E|=0$ and $|V|-1=1-1=0$, we have $|E| \geq |V|-1$
- Inductive hypothesis
  - If graph has $|V|=k$ vertices then $|E| \geq k-1$
- Inductive step
  - Let G be any connected graph with $|V|=k+1$ vertices
  - We must show that $|E| \geq k$

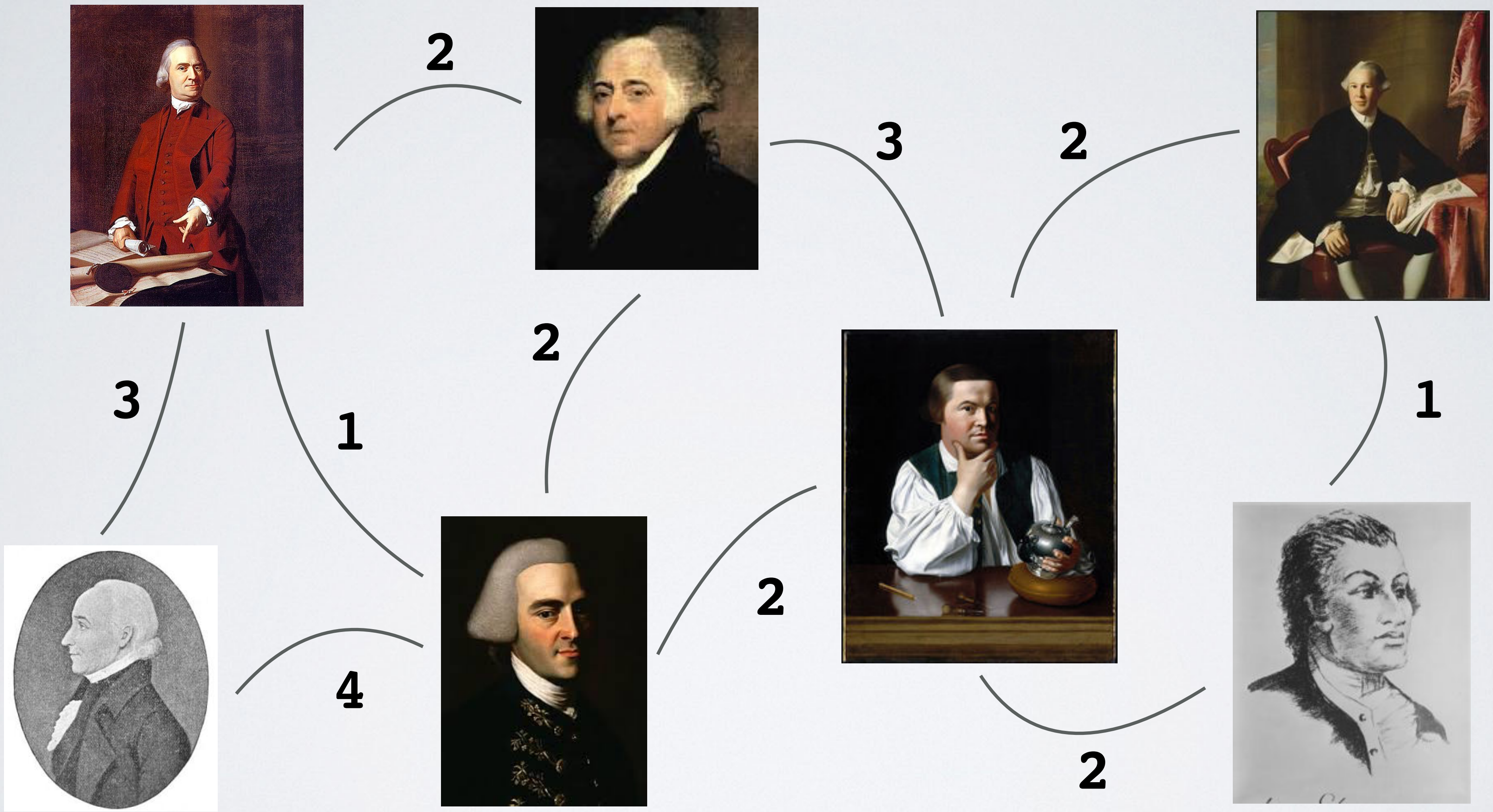
Graph Proof 2

- Inductive step
  - Let `G` be any connected graph with `|V|=k+1` vertices
  - We must show that `|E| ≥ k`
- Let `u` be the vertex of minimum degree in `G`
  - `deg(u) ≥ 1` since `G` is connected
- If `deg(u) = 1`
  - Let `G'` be `G` without `u` and its `1` incident edge
  - `G'` has `k` vertices because we removed `1` vertex from `G`
  - `G'` is still connected because we only removed a leaf
  - So by inductive hypothesis, `G'` has at least `k–1` edges
  - which means that `G` has at least `k` edges

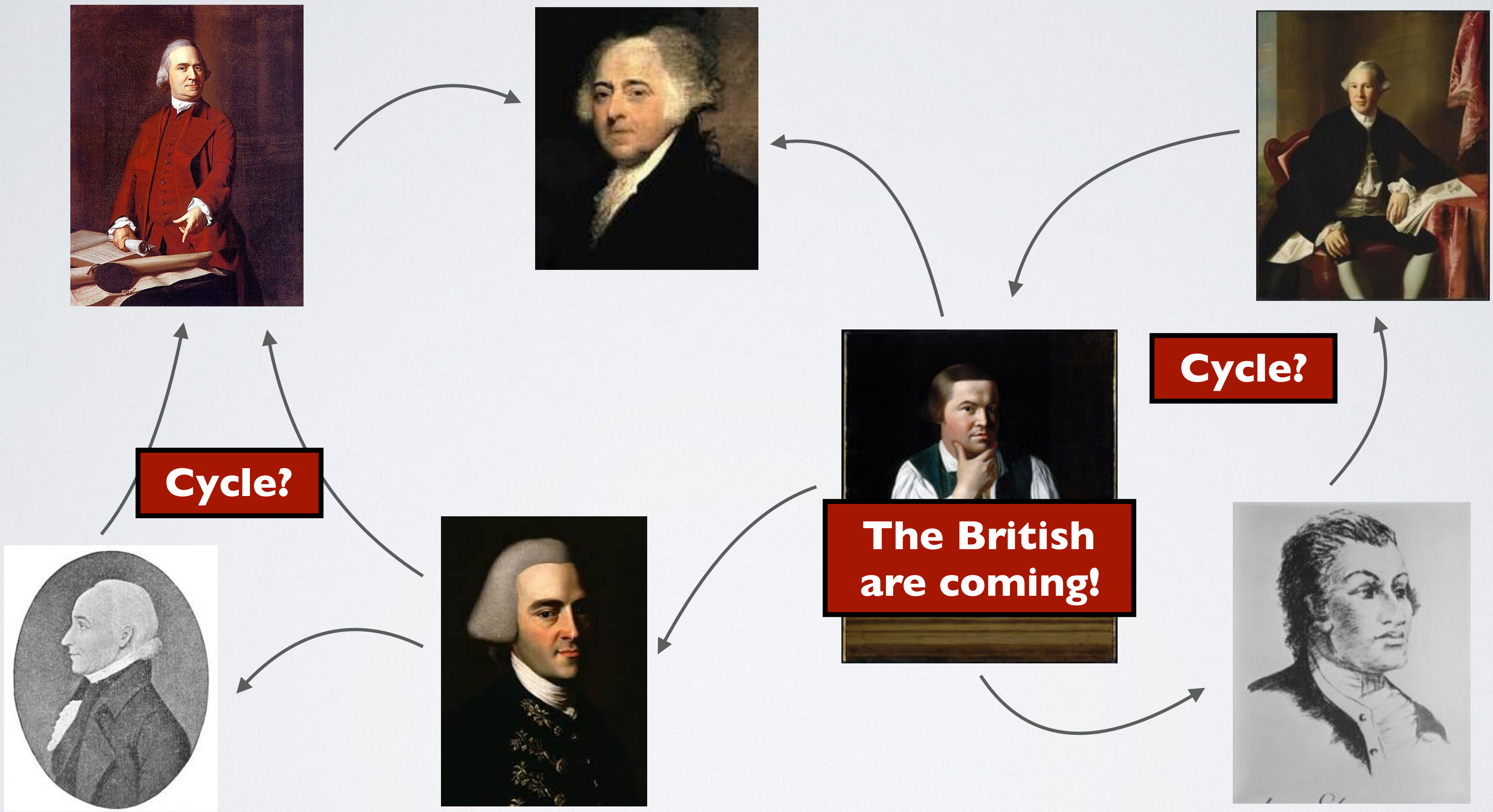
Graph Proof 2

- If $\mathtt{deg(u)} \geq 2$
  - Every vertex has at least two incident edges
  - So the total degree $\mathtt{D}$ of the graph is $\mathtt{D} \geq 2(\mathtt{k}+1)$
  - But we know from the last proof that $\mathtt{D}=2|\mathtt{E}|$
    - so $2|\mathtt{E}| \geq 2(\mathtt{k}+1) \implies |\mathtt{E}| \geq \mathtt{k}+1 \implies |\mathtt{E}| \geq \mathtt{k}$
- We showed it is true for $|\mathtt{V}|=1$ (base case)…
  - …and for $|\mathtt{V}|=\mathtt{k}+1$ assuming it is true for $|\mathtt{V}|=\mathtt{k}$…
  - …so it is true for all $|\mathtt{V}| \geq 1$

Undirected graph



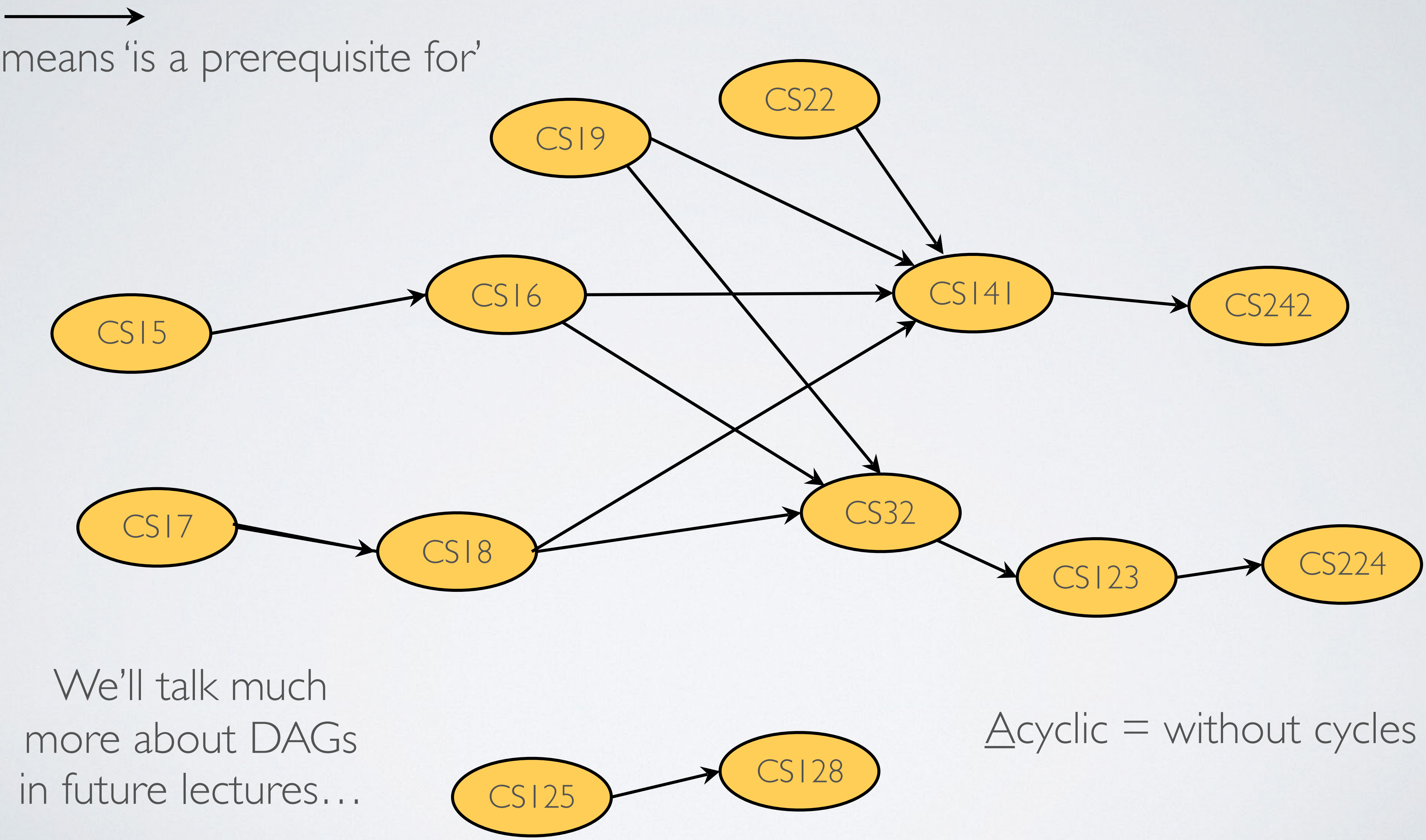
Directed graph



Edge Types

- Undirected edge
  - *unordered* pair of vertices (L,R)
- Directed edge
  - *ordered* pair of vertices (L,R)
  - first vertex L is the origin
  - second vertex R is the destination

Directed Acyclic Graph (DAG)



We'll talk much more about DAGs in future lectures…

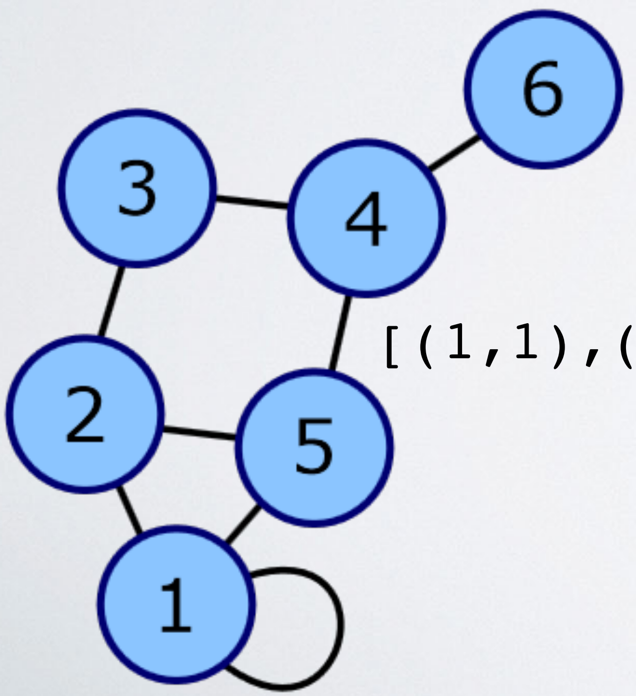
Acyclic = without cycles

Graph Representations

- Vertices usually stored in a List or Set
- 3 common ways of representing which vertices are adjacent
  - Edge list (or set)
  - Adjacency lists (or sets)
  - Adjacency matrix

Edge List

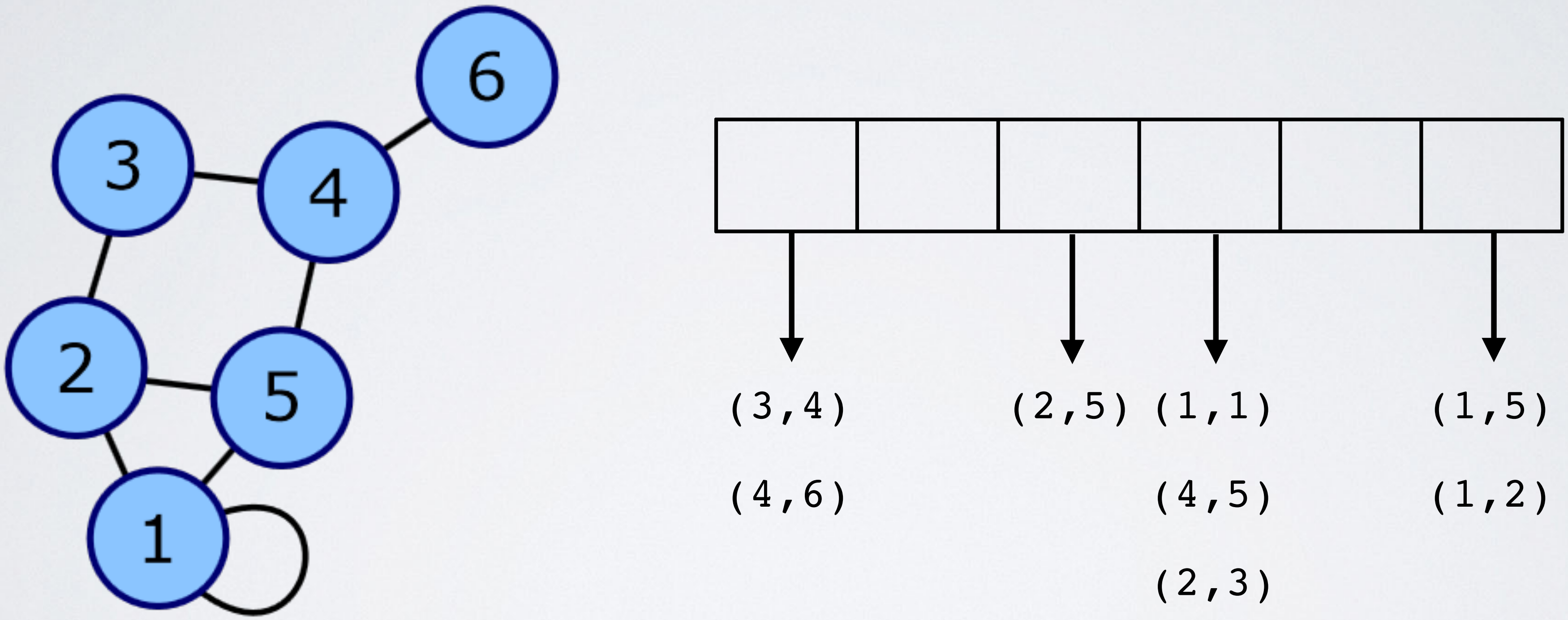
- Represents adjacencies as a list of pairs
- Each element of list is a single edge **`(a,b)`**
- Since the order of list doesn't matter
  - can use hashset to improve runtime of adjacency testing



`[(1,1),(1,2),(1,5),(2,3),(2,5),(3,4),(4,5),(4,6)]`

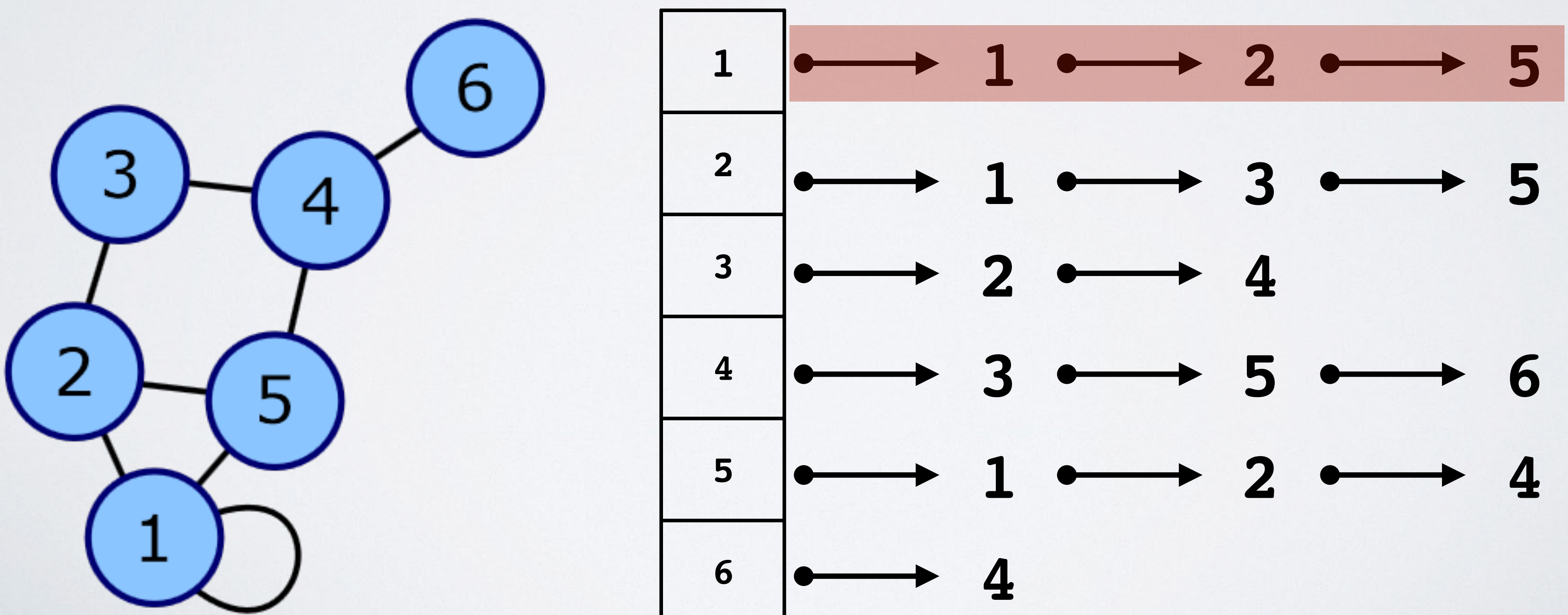
Edge Set

- Store all the edges in a Hashset



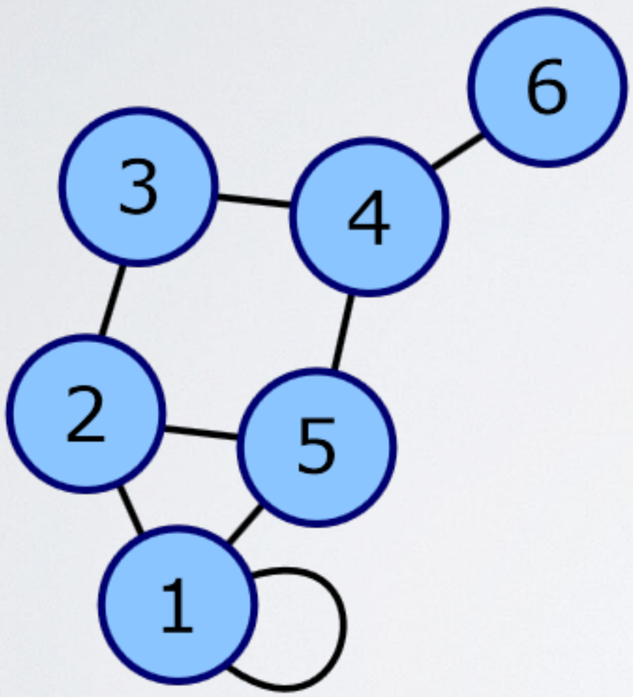
Adjacency Lists

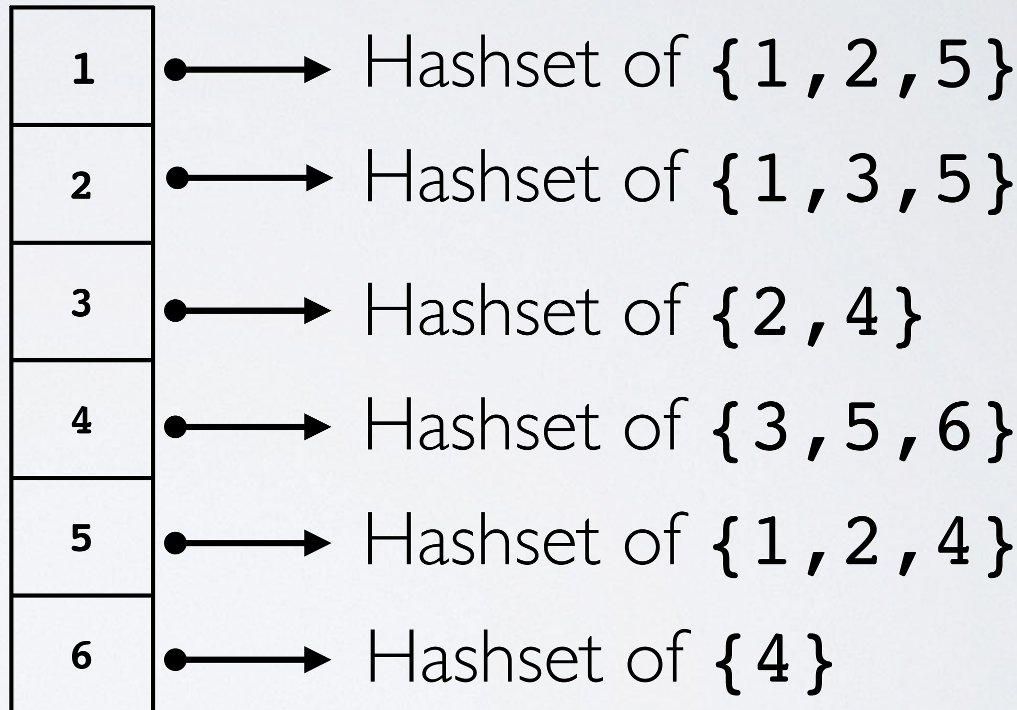
- Each vertex has an associated list with its neighbors
- Since the order of elements in lists doesn't matter
  - lists can be hashsets instead



Adjacency Set

- Each vertex associated Hashset of its neighbors

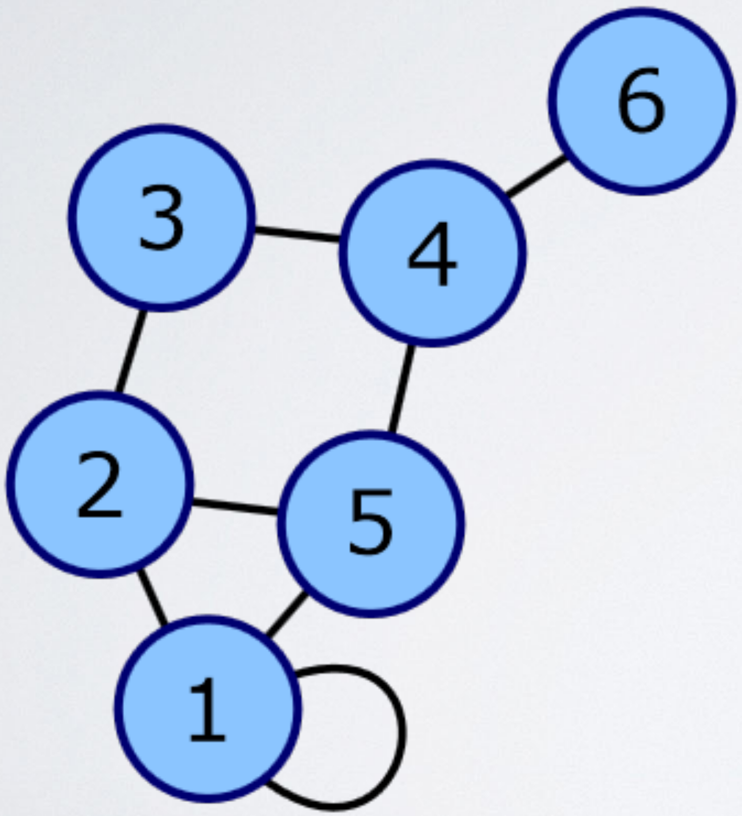




Adjacency Matrix

- Matrix with **`n`** rows and **`n`** columns
  - **`n`** is number of vertices
  - If **`u`** is adjacent to **`v`** then `M[u,v]=T`
  - If **`u`** is not adjacent to **`v`** then `M[u,v]=F`
- If graph is undirected then `M[u,v]=M[v,u]`

Adjacency Matrix



	1	2	3	4	5	6
1	T	T	F	F	T	F
2	T	F	T	F	T	F
3	F	T	F	T	F	F
4	F	F	T	F	T	T
5	T	T	F	T	F	F
6	F	F	F	T	F	F

Adjacency Matrix

- Initialize matrix to predicted size of graph
  - we can always expand later
- When vertex is added to graph
  - reserve a row and column of matrix for that vertex
- When vertex is removed
  - set its entire row and column to false
- Since we can't remove rows/columns from arrays
  - keep separate collection of vertices that are actually present in graph

Graph ADT

- Vertices and edges can store values
  - Ex: edge weights
- Accessor methods
  - **vertices**( )
  - **edges**( )
  - **incidentEdges**(vertex)
  - **areAdjacent**(v_1, v_2)
  - **endVertices**(edge)
  - **opposite**(vertex, edge)
- Update methods
  - **insertVertex**(value)
  - **insertEdge**(v_1, v_2)
    - sometimes this function also takes a value
so **insertEdge**(v_1, v_2,val)
  - **removeVertex**(vertex)
  - **removeEdge**(edge)

Big-O Performance

Activity #1

3 min

Big-O Performance

Activity #1

3 min

Big-O Performance

Activity #1

2 min

Big-O Performance

Activity #1

1 min

Big-O Performance

Activity #1

0 min

Big-O Performance

	Edge Set	Adjacency Sets	Adjacency Matrix
Overall Space[1]	O(|V| + |E|)	O(|V| + |E|)	O(|V|²)
vertices( )[1]	O(1)*	O(1)*	O(1)*
edges( )	O(1)*	O(|E|)	O(|V|²)
incidentEdges(v)	O(|E|)	O(1)*	O(|V|)
areAdjacent (v_1, v_2)	O(1)	O(1)	O(1)
insertVertex(v)	O(1)	O(1)	O(|V|)
insertEdge(v_1, v_2)	O(1)	O(1)	O(1)
removeVertex(v)	O(|E|)	O(|V|)	O(|V|)
removeEdge(v_1, v_2)	O(1)	O(1)	O(1)

[1] In all approaches, we maintain an additional list or set of vertices

* in place (return pointer)

Big-O Performance (Edge Set)

Operation	Runtime	Explanation
vertices()	`O(1)`	Return set of vertices
edges()	`O(1)`	Return set of edges
incidentEdges(v)	`O(|E|)`	Iterate through each edge and check if it contains vertex **`v`**
areAdjacent(v_1,v_2)	`O(1)`	Check if **`(`**v_1**`,`**v_2**`)`** exists in the set
insertVertex(v)	`O(1)`	Add vertex **`v`** to the vertex list
insertEdge(v_1,v_2)	`O(1)`	Add element **`(`**v_1**`,`**v_2**`)`** to the set
removeVertex(v)	`O(|E|)`	Iterate through each edge and remove it if it has vertex **`v`**
removeEdge(v_1,v_2)	`O(1)`	Remove edge **`(`**v_1**`,`**v_2**`)`**

Big-O Performance (Adjacency Set)

Operation	Runtime	Explanation
vertices()	`O(1)`	Return the set of vertices
edges()	`O(|E|)`	Concatenate each vertex with its subsequent vertices
incidentEdges(v)	`O(1)`	Return **v**'s edge set
areAdjacent(v_1,v_2)	`O(1)`	Check if $\mathbf{v_2}$ is in $\mathbf{v_1}$'s set
insertVertex(v)	`O(1)`	Add vertex **v** to the vertex set
insertEdge(v_1,v_2)	`O(1)`	Add $\mathbf{v_1}$ to $\mathbf{v_2}$'s edge set and vice versa
removeVertex(v)	`O(|V|)`	Remove **v** from each of its adjacent vertices' sets and remove **v**'s set
removeEdge(v_1,v_2)	`O(1)`	Remove $\mathbf{v_1}$ from $\mathbf{v_2}$'s set and vice versa

Big-O Performance (Adjacency Matrix)

Operation	Runtime	Explanation
vertices()	`O(1)`	Return the set of vertices
edges()	`O(|V|²)`	Iterate through the entire matrix
incidentEdges(v)	`O(|V|)`	Iterate through v's row or column to check for trues Note: row/col are the same in an undirected graph.
areAdjacent(v_1,v_2)	`O(1)`	Check index (v_1,v_2) for a true
insertVertex(v)	`O(|V|)*`	Add vertex v to the matrix (* `O(1)` amortized)
insertEdge(v_1,v_2)	`O(1)`	Set index (v_1,v_2) to true
removeVertex(v)	`O(|V|)`	Set v's row and column to false and remove v from the vertex list
removeEdge(v_1,v_2)	`O(1)`	Set index (v_1,v_2) to false

BFT and DFT

- Remember BFT and DFT on trees?
- We can also do them on graphs
  - a tree is just a special kind of graph
  - often used to find certain values in graphs

BFT/DFT on Graphs

Activity #2

1 min

BFT/DFT on Graphs

Activity #2

1 min

BFT/DFT on Graphs

Activity #2

0 min

Breadth First Traversal: Tree vs. Graph

```
function treeBFT(root):
   //Input: Root node of tree
   //Output: Nothing
   Q = new Queue()
   Q.enqueue(root)
   while Q is not empty:
      node = Q.dequeue()
      doSomething(node)
      enqueue node's children
```

doSomething( ) could print, add to list, decorate node etc…

```
function graphBFT(start):
   //Input: start vertex
   //Output: Nothing
   Q = new Queue()
   start.visited = true
   Q.enqueue(start)
   while Q is not empty:
      node = Q.dequeue()
      doSomething(node)
      for neighbor in adj nodes:
         if not neighbor.visited:
            neighbor.visited = true
            Q.enqueue(neighbor)
```

Mark nodes as visited otherwise you will loop forever!

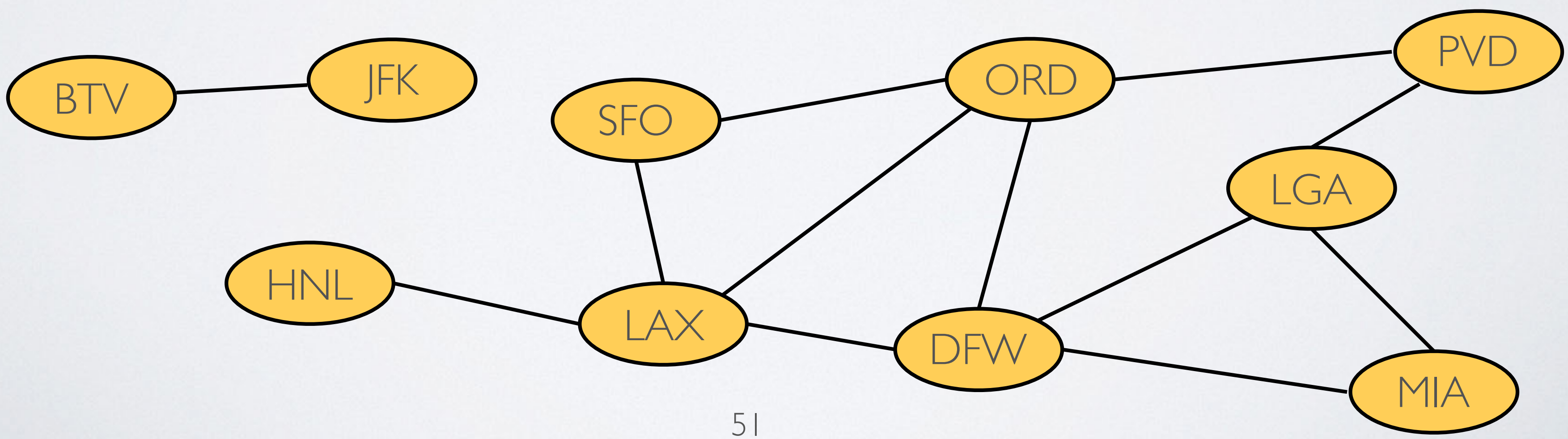
Depth First Traversal

- To do DFT on graph, replace queue with stack
- Can also be done recursively

```
function recursiveDFT(node):
   // Input: start node
   // Output: Nothing
   node.visited = true
   for neighbor in node's adjacent vertices:
      if not neighbor.visited:
         recursiveDFT(neighbor)
```

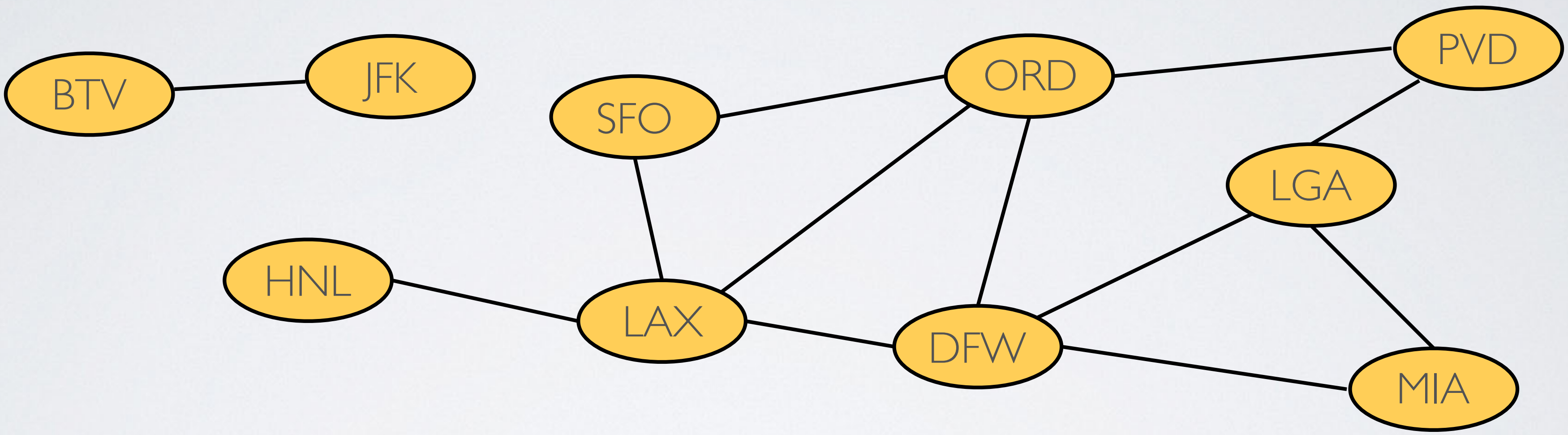
Applications: Flight Paths Exist

- Given undirected graph with airports & flights
  - is it possible to fly from one airport to another?
- Strategy
  - use breadth first search starting at first node
  - and determine if ending airport is ever visited



Applications: Flight Paths Exist

- Is there flight from SFO to PVD?



Applications: Flight Paths Exist

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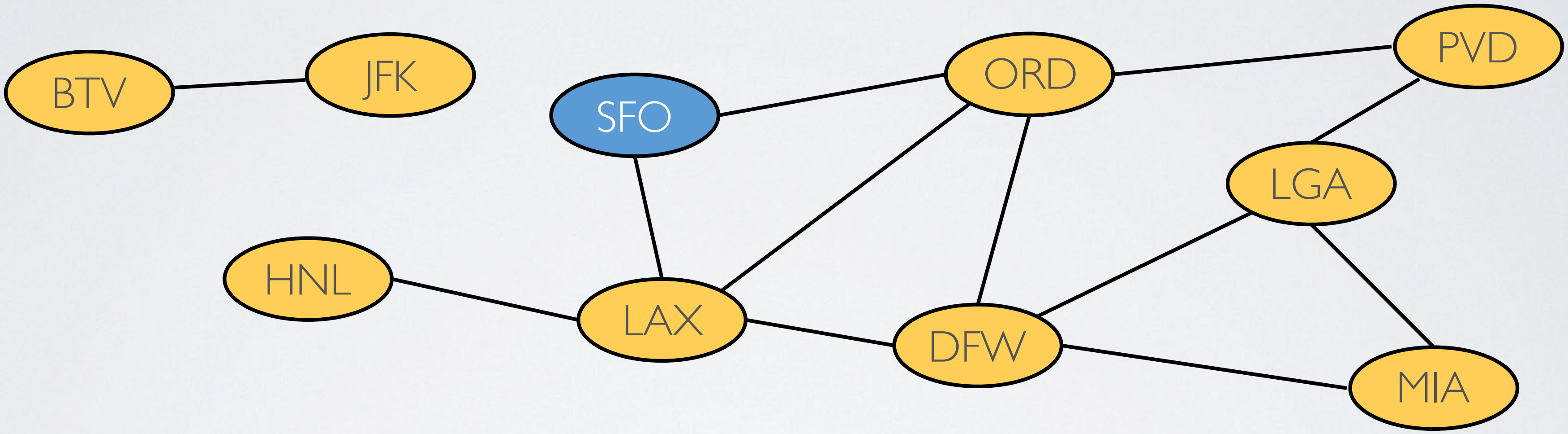

Applications: Flight Paths Exist

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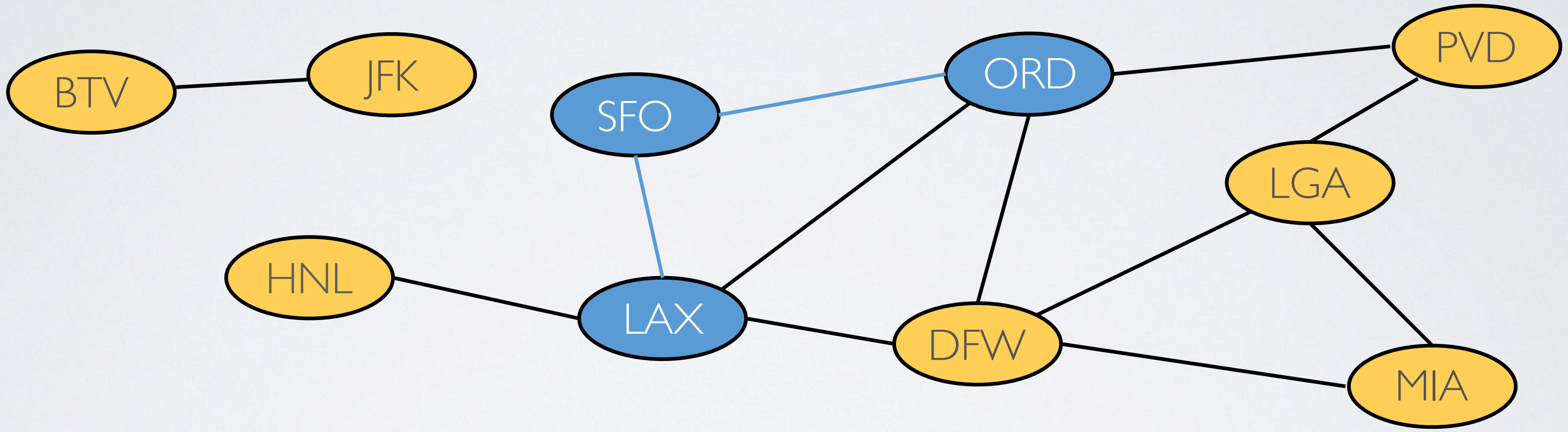
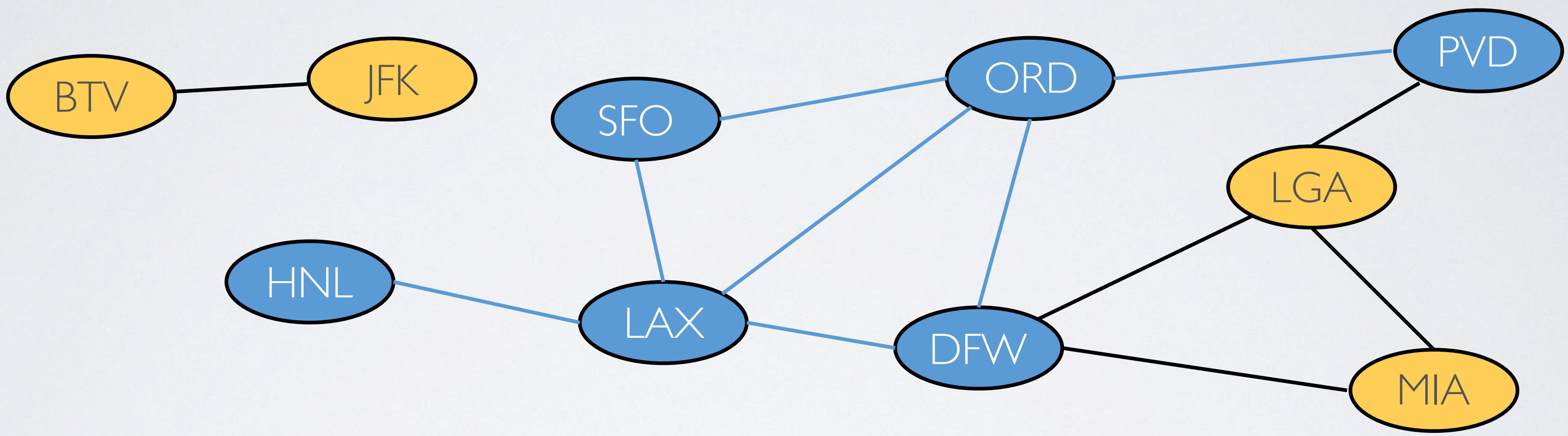


Applications: Flight Paths Exist

- Is there flight from SFO to PVD?



- Yes! but how do we do it with code?

Flight Paths Exist Pseudo-Code

```
function pathExists(from, to):
   //Input: from: vertex, to: vertex
   //Output: true if path exists, false otherwise
   Q = new Queue()
   from.visited = true
   Q.enqueue(from)
   while Q is not empty:
      airport = Q.dequeue()
      if airport == to:
         return true
      for neighbor in airport's adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            Q.enqueue(neighbor)
   return false
```

Applications: Flight Layovers

- Given undirected graph with airports & flights
  - decorate vertices w/ least number of stops from a given source
  - if no way to get to a an airport decorate w/ ∞
- Strategy
  - decorate each node w/ initial 'stop value' of ∞
  - use breadth first search to decorate each node…
  - …w/ 'stop value' of one greater than its previous value

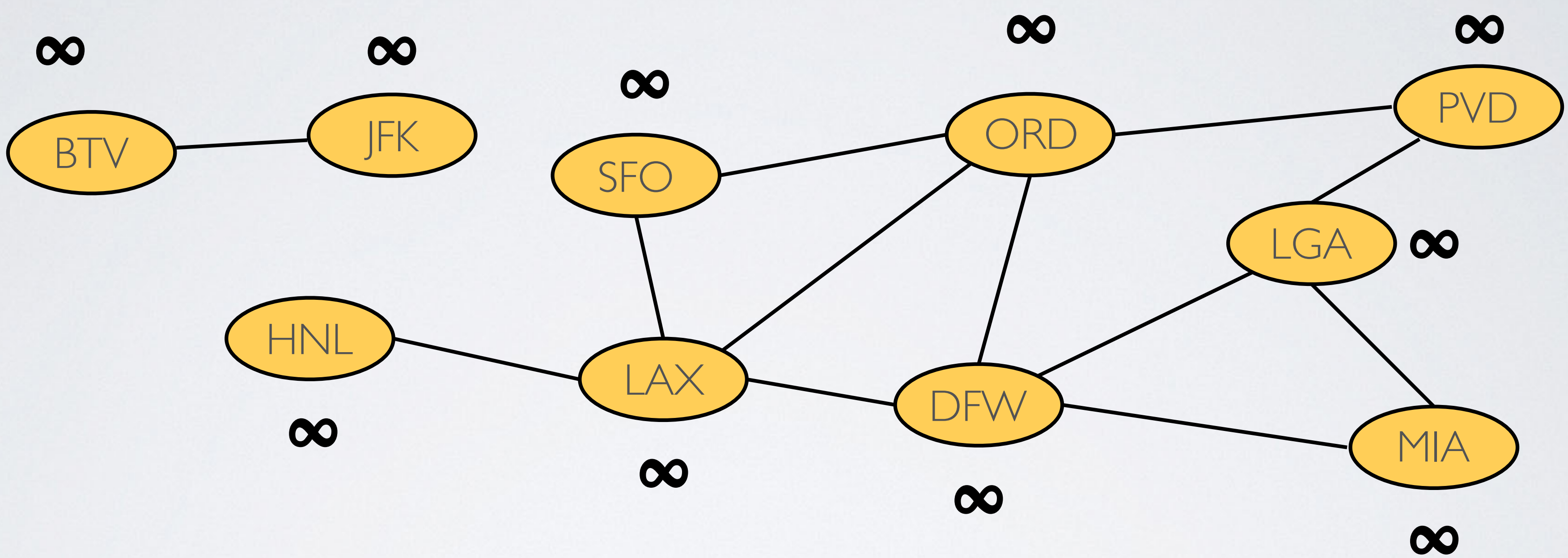
Flight Layovers Pseudo-Code

```
function numStops(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest number of
   //         layovers from source.

   for every node in G:
      node.stops = infinity

   Q = new Queue()
   source.stops = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      airport = Q.dequeue()
      for neighbor in airport’s adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            neighbor.stops = airport.stops + 1
            Q.enqueue(neighbor)
```

Flight Layovers Pseudo-Code



Flight Layovers Pseudo-Code