Outline

- Medians
- Selection
- Randomized Selection
- Median-of-Medians Selection
Medians

- The median of a collection of numbers
  - is the middle element
  - half of the numbers smaller and half larger

- Used to summarize the collection

- The mean or average can also be used...
  - ...but averages can be corrupted by outliers

- What are the mean & median of

  - \([9, 5, 4, 6, 5, 7, 10000, 6, 4, 8]\)
    - mean 1005.4 & median 6

- Finding median is easy: sort list and pick middle element
  - \(O(n \log n)\)...can we do better?
Medians

- If we can find median quickly we could use it in Quicksort
- Recall that Quicksort picks pivot at random
  - if pivot is max or min element every time
  - runtime is $O(n^2)$
- If we could find pivot in $O(n)$
  - Quicksort would have $O(n \log n)$ worst-case runtime
  - Same as Merge Sort
Selection

- Let's consider a more general problem
- Selection
  - given a list $L$ and a integer $k$
  - output the $k$th smallest element in list
- Median with selection
  - Selection with $k = n/2$
Selection

- Divide and conquer
  - divide: pick random element (called pivot) and partition set into
    - L: elements less than x
    - E: elements equal to x
    - G: elements larger than x
  - recur:
    - if \( k \leq |L| \): call \texttt{select}(L, k)
    - if \( |L| < k \leq |L| + |E| \): return x
    - if \( k > |L| + |E| \): call \texttt{select}(G, k - (|L| + |E|))
  - conquer: return
Hoare's Selection Pseudo-Code

```python
select(list, k):
    # Base case omitted
    pivot = list[rand(0, list.size)]
    L = []    E = []    G = []
    for x in list:
        if x < pivot: L.append(x)
        if x == pivot: E.append(x)
        if x > pivot: G.append(x)
    if k <= L.size:
        return select(L, k)
    else if k <= (L.size + E.size):
        return pivot
    else
        return select(G, k – (L.size + E.size))
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Activity #1+2
Selection

```python
select(list, k):
    // Base case omitted
    pivot = list[rand(0, list.size)]
    L = []  E = []  G = []
    for x in list:
        if x < pivot: L.append(x)
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        if x > pivot: G.append(x)
    if k <= L.size:
        return select(L, k)
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        return pivot
    else
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Selection Analysis

- When selection uses random pivot...
  - ...it is known as Hoare's selection
- How fast is Hoare's selection?
  - like quicksort it has $O(n^2)$ run time if we pick the wrong pivots
- We consider expected runtime since algorithm is randomized
- We assume all elements are distinct (worst-case scenario)
  - if list has more than one copy of pivot,
  - it would shrink the sub-lists and improve runtime
Selection Analysis

- The recurrence relation of selection has form
  \[ T(n) = \frac{1}{n} \cdot \sum_{i=0}^{n-1} (???) + O(n) \]
- Each pivot has equal probability of being chosen
- Each pivot splits list in two lists of size \( i \) and \( n-1-i \)
- Algorithm can recurse on either sub-list
  - \( k \)th smallest element could be anywhere...
  - …so it will recurse on first sub-list w/ prob \( i/(n-1) \)
  - …and second sub-list w/ prob \( (n-1-i)/(n-1) \)
Selection Analysis

- Recurrence relation now has form

\[
T(n) = \frac{1}{n} \cdot \sum_{i=0}^{n-1} \left( \frac{i}{n-1} \cdot T(i) + \frac{n-1-i}{n-1} \cdot T(n-1-i) \right) + O(n)
\]

- which is \(O(n)\) (see Wocjan for proof)
Selection in Quicksort

- Selection runtime
  - Expected $O(n)$
  - Worst-case is still $O(n^2)$
- If we use selection in Quicksort
  - we still have $O(n \log n)$ expected time
  - and $O(n^2)$ worst-case
- But we’re not done!
Outline

- Medians
- Selection
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Median-of-Medians Select

- General Strategy
  - pick pivot that is “always good”
  - i.e., between 25th and 75th percentile
- Do this by picking the median of medians (mom)
  - partition list into $\frac{n}{5}$ lists of size 5
  - sort each list in $O(1)$ time
  - choose the median of each list
  - call mom-select recursively on list of $\frac{n}{5}$ medians
- Use the mom as pivot and continue w/ selection algorithm
Median-of-Medians Select

```python
momSelect(list, k)
    // Base case omitted
    miniLists = divide list into n/5 lists of 5
    medians = []
    for miniList in miniLists:
        sort5(miniList) // in O(1) b/c miniList always size 5
        medians.append(miniList[2])
    pivot = momSelect(medians, medians.size/2)

    L = []   E = []   G = []
    for x in list:
        if x < pivot: L.append(x)
        if x == pivot: E.append(x)
        if x > pivot: R.append(x)
    if k <= L.size:
        return momSelect(L, k)
    else if k <= (L.size + E.size):
        return pivot
    else
        return momSelect(G, k - (L.size + E.size))
```
Median-of-Medians Select

- Sorting a list of numbers from 1 to 50

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<th>4</th>
<th>3</th>
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<td>39</td>
<td>41</td>
<td>46</td>
<td>48</td>
<td></td>
</tr>
</tbody>
</table>

- Guaranteed to be larger than red numbers and larger than blue numbers which is between 25th and 75th percentiles. Great pivot choice!

- How many elements will pivot eliminate? What is area of blue or red region?
  - height: 3
  - width: \((n/5)/2\)
  - area: \(3n/10\)

- leaves problem of size at most \(7n/10\)
Median-of-Medians Select

Activity #3
Median-of-Medians Select

3 min

Activity #3
Median-of-Medians Select

2 min

Activity #3
Median-of-Medians Select

1 min

Activity #3
Median-of-Medians Select

Activity #3
Median-of-Medians Select

- With median-of-medians as pivot...
- ...the select recurrence relation is:

\[
T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)
\]

From recursing on list of \(n/5\) medians to find median of median

From recursing on list guaranteed to be at most \(7/10\) size of original

- This is \(O(n)\) (see Epstein for proof)
Summary

- We can perform select in worst-case $O(n)$
- ...which means we can find medians in worst-case $O(n)$
- ...which means we can perform Quicksort in worst-case $O(n \log n)$
- In practice, Quicksort w/ random pivots is faster
- Quicksort w/ median of medians is useful when worst-case performance is crucial
- We don’t expect you to implement median of medians when implementing Quicksort
Readings

- Dasgupta et al.
  - Section 2.4: analysis of median finding algorithms
- Wocjan’s analysis of Selection w/ random pivot
  - http://www.eecs.ucf.edu/courses/cot5405/fall2010/chapter1_2/QuickSelAvgCase.pdf
- Proof that median of median is $O(n)$ (Epstein reading)