Breadth-first Search and Shortest Paths in Graphs

CS16: Introduction to Data Structures & Algorithms

Summer 2021

BFT and DFT

- Remember BFT and DFT on trees?
- We can also do them on graphs
  - a tree is just a special kind of graph
  - often used to find certain values in graphs

Breadth First Traversal: Tree vs. Graph

```
function treeBFT(root):
   //Input: Root node of tree
   //Output: Nothing
   Q = new Queue()
   Q.enqueue(root)
   while Q is not empty:
      node = Q.dequeue()
      doSomething(node)
      enqueue node's children
```

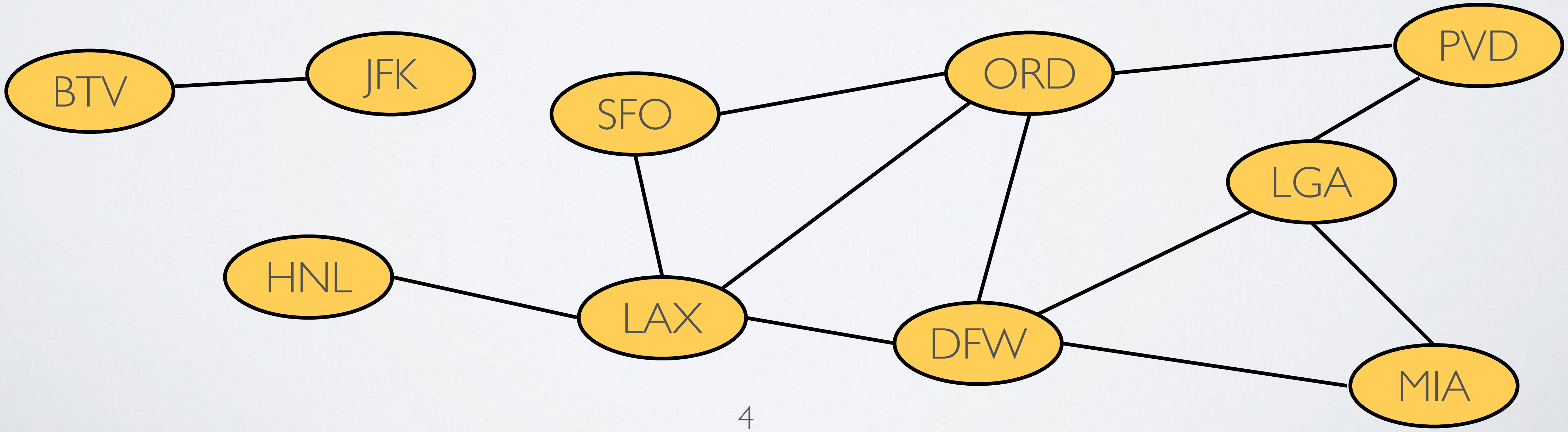
doSomething( ) could print, add to list, decorate node etc…

```
function graphBFT(start):
   //Input: start vertex
   //Output: Nothing
   Q = new Queue()
   start.visited = true
   Q.enqueue(start)
   while Q is not empty:
      node = Q.dequeue()
      doSomething(node)
      for neighbor in adj nodes:
         if not neighbor.visited:
            neighbor.visited = true
            Q.enqueue(neighbor)
```

Mark nodes as visited otherwise you will loop forever!

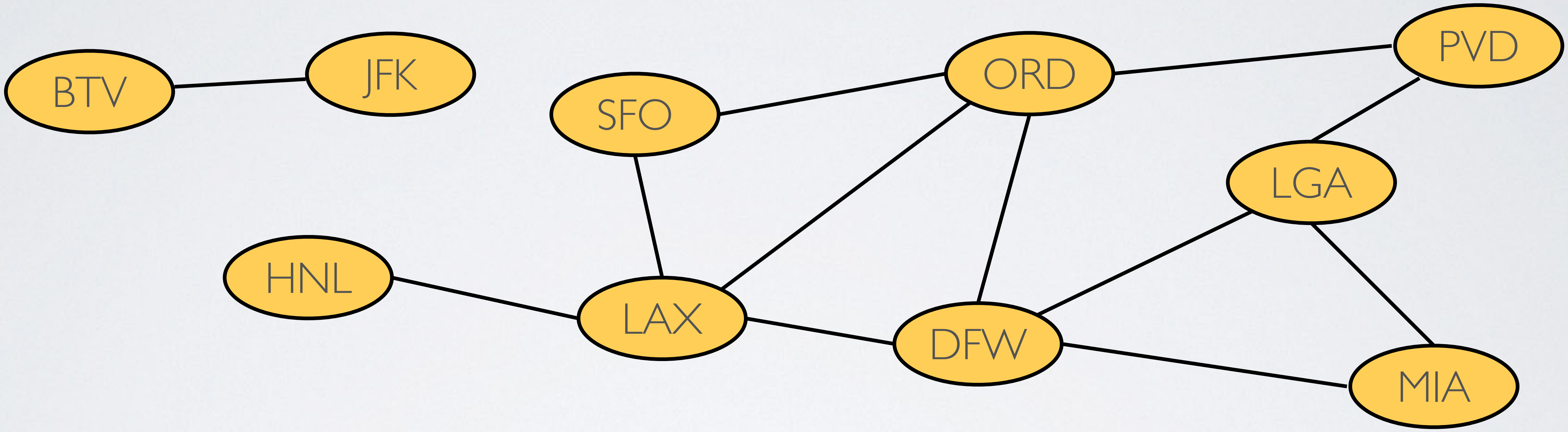
Applications: Flight Paths Exist

- Given undirected graph with airports & flights
  - is it possible to fly from one airport to another?
- Strategy
  - use breadth first search starting at first node
  - and determine if ending airport is ever visited



Applications: Flight Paths Exist

- Is there flight from SFO to PVD?



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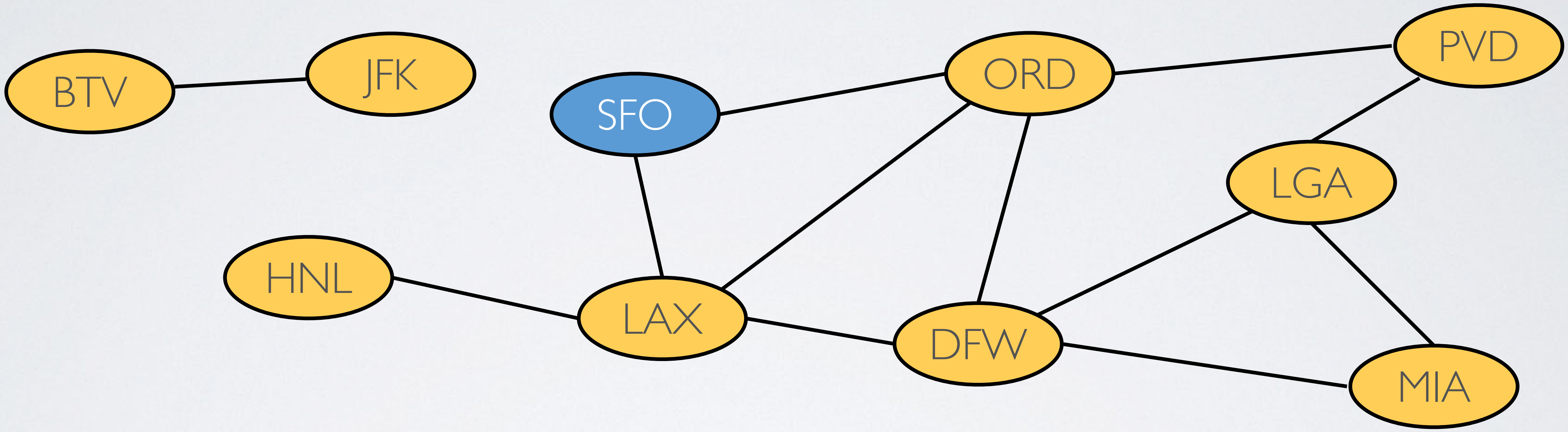
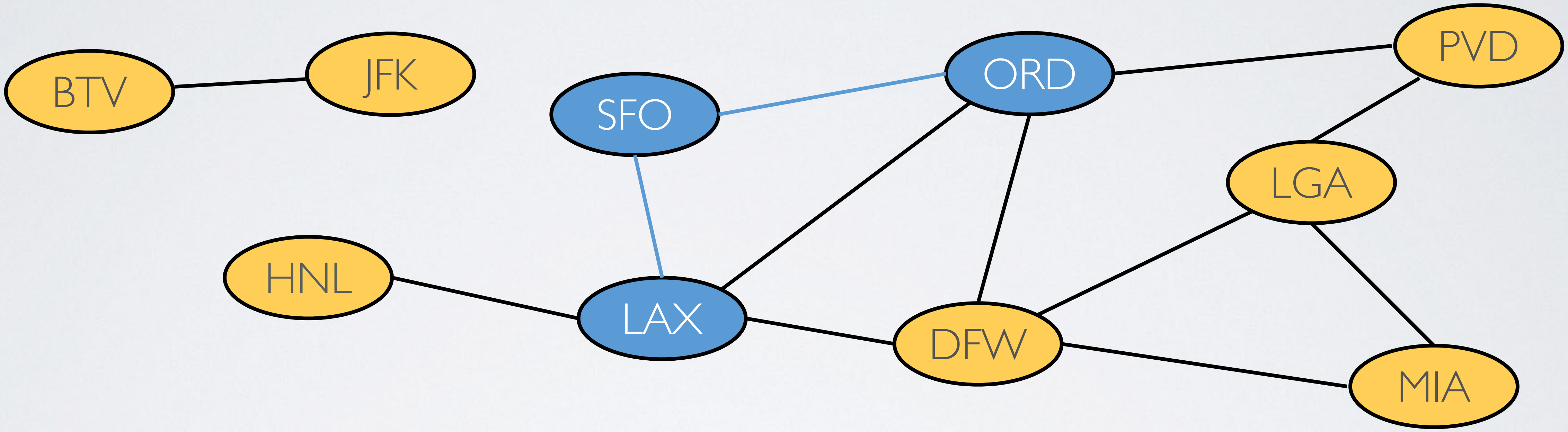


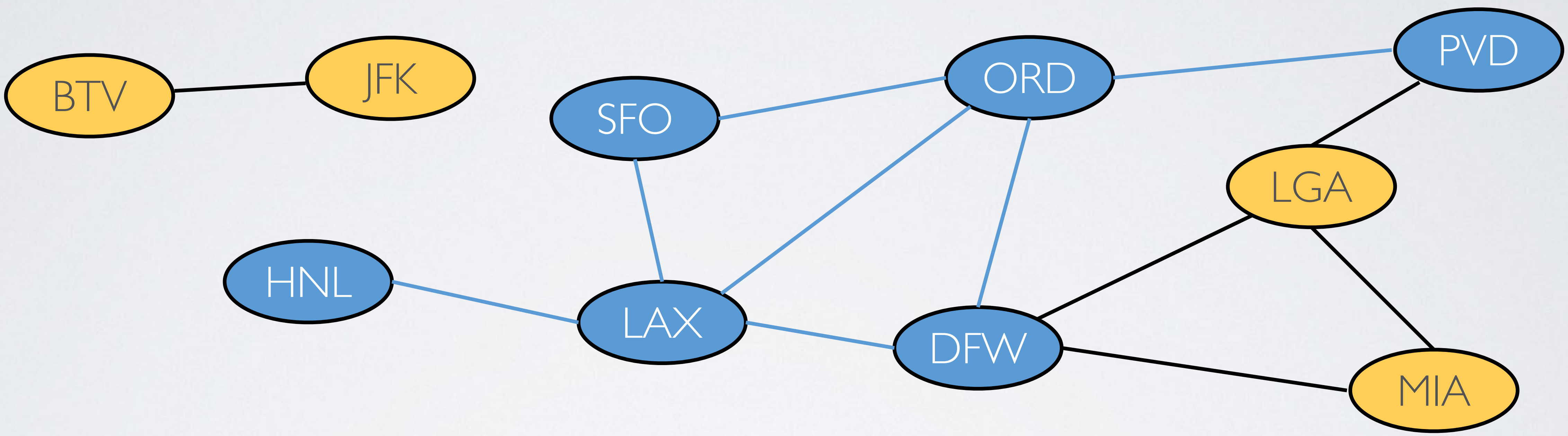
Applications: Flight Paths Exist

- Is there flight from SFO to PVD?



Applications: Flight Paths Exist

- Is there flight from SFO to PVD?



- Yes! but how do we do it with code?

Flight Paths Exist Pseudo-Code

```
function pathExists(from, to):
   //Input: from: vertex, to: vertex
   //Output: true if path exists, false otherwise
   Q = new Queue()
   from.visited = true
   Q.enqueue(from)
   while Q is not empty:
      airport = Q.dequeue()
      if airport == to:
         return true
      for neighbor in airport's adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            Q.enqueue(neighbor)
   return false
```

Applications: Flight Layovers

- Given undirected graph with airports & flights
  - decorate vertices w/ least number of stops from a given source
  - if no way to get to a an airport decorate w/ ∞
- Strategy
  - decorate each node w/ initial 'stop value' of ∞
  - use breadth first traversal to decorate each node…
  - …w/ 'stop value' of one greater than its previous value

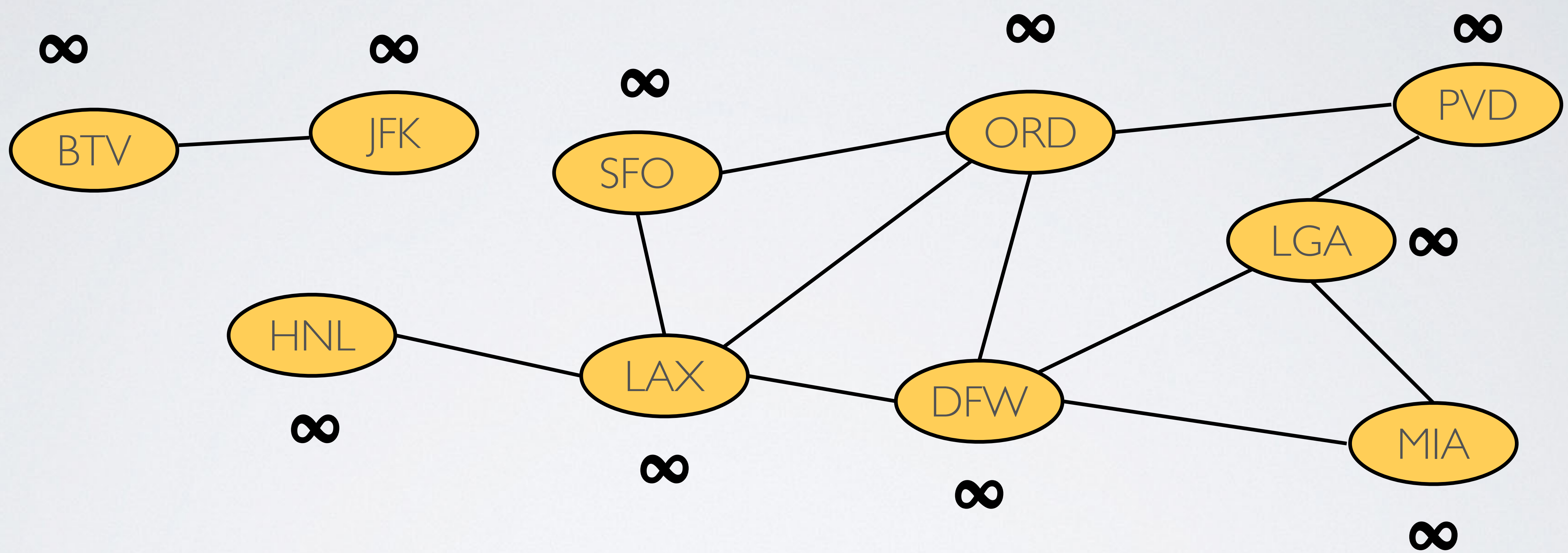
Flight Layovers Pseudo-Code

```
function numStops(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest number of
   //         layovers from source.

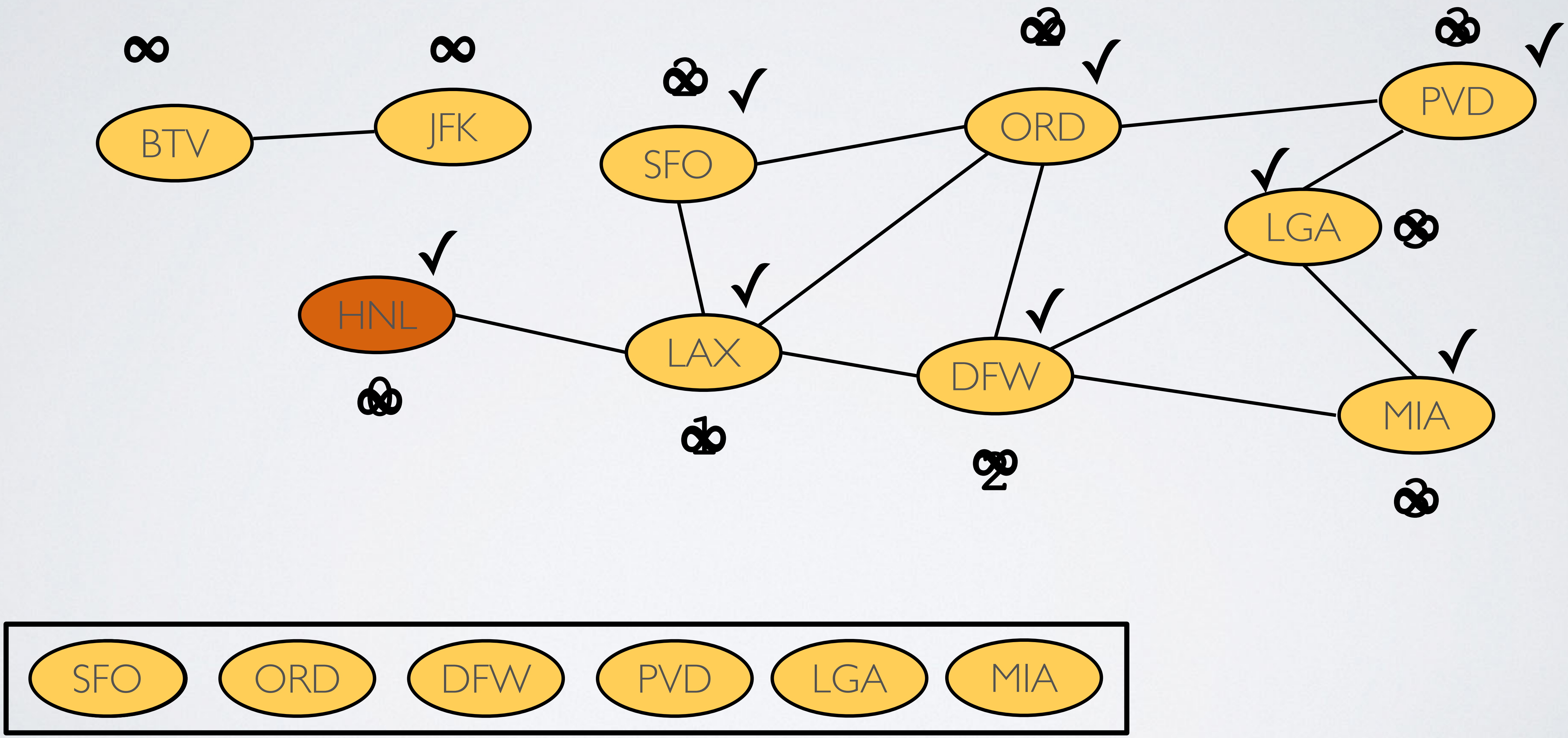
   for every node in G:
      node.stops = infinity

   Q = new Queue()
   source.stops = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      airport = Q.dequeue()
      for neighbor in airport's adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            neighbor.stops = airport.stops + 1
            Q.enqueue(neighbor)
```

Flight Layovers Example



Flight Layovers Example



What if we want a path?

- numStops gives us the distance
- Want to know *how* to get from (e.g.) HNL to LGA
- Strategy: at each node we reach, record the node we used to get there

Flight Layovers Pseudo-Code

```
function numStops(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest number of
   //         layovers from source.

   for every node in G:
      node.stops = infinity
      node.previous = null

   Q = new Queue()
   source.stops = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      airport = Q.dequeue()
      for neighbor in airport's adjacent nodes:
         if not neighbor.visited:
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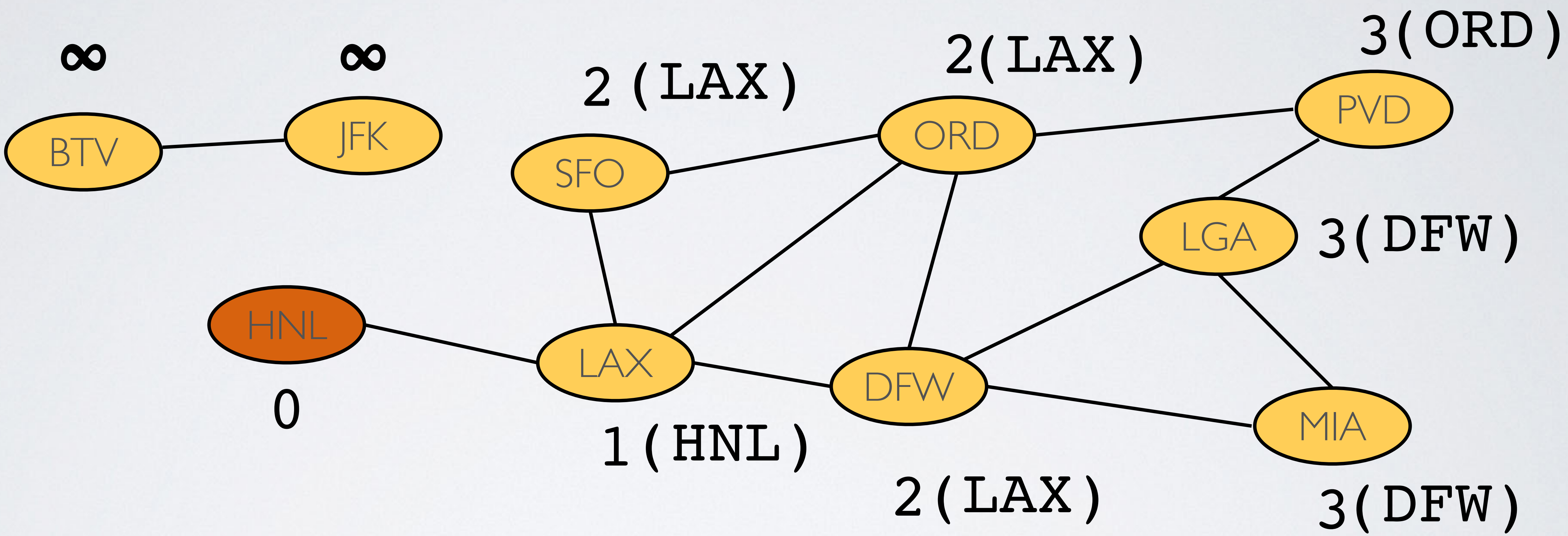
            neighbor.stops = airport.stops + 1
            neighbor.previous = airport
            Q.enqueue(neighbor)
```

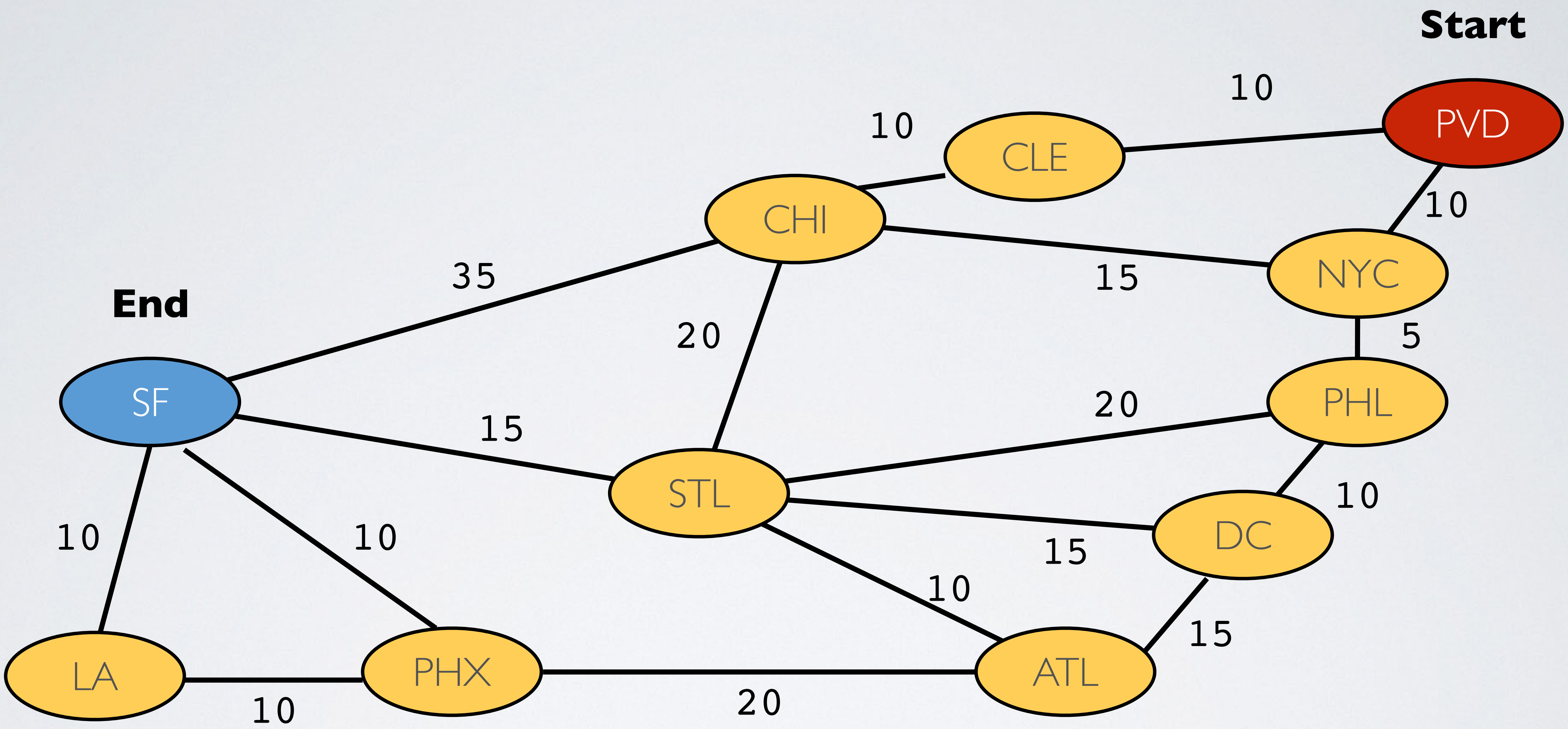
Flight paths



Single **S**ource **S**hortest **P**aths

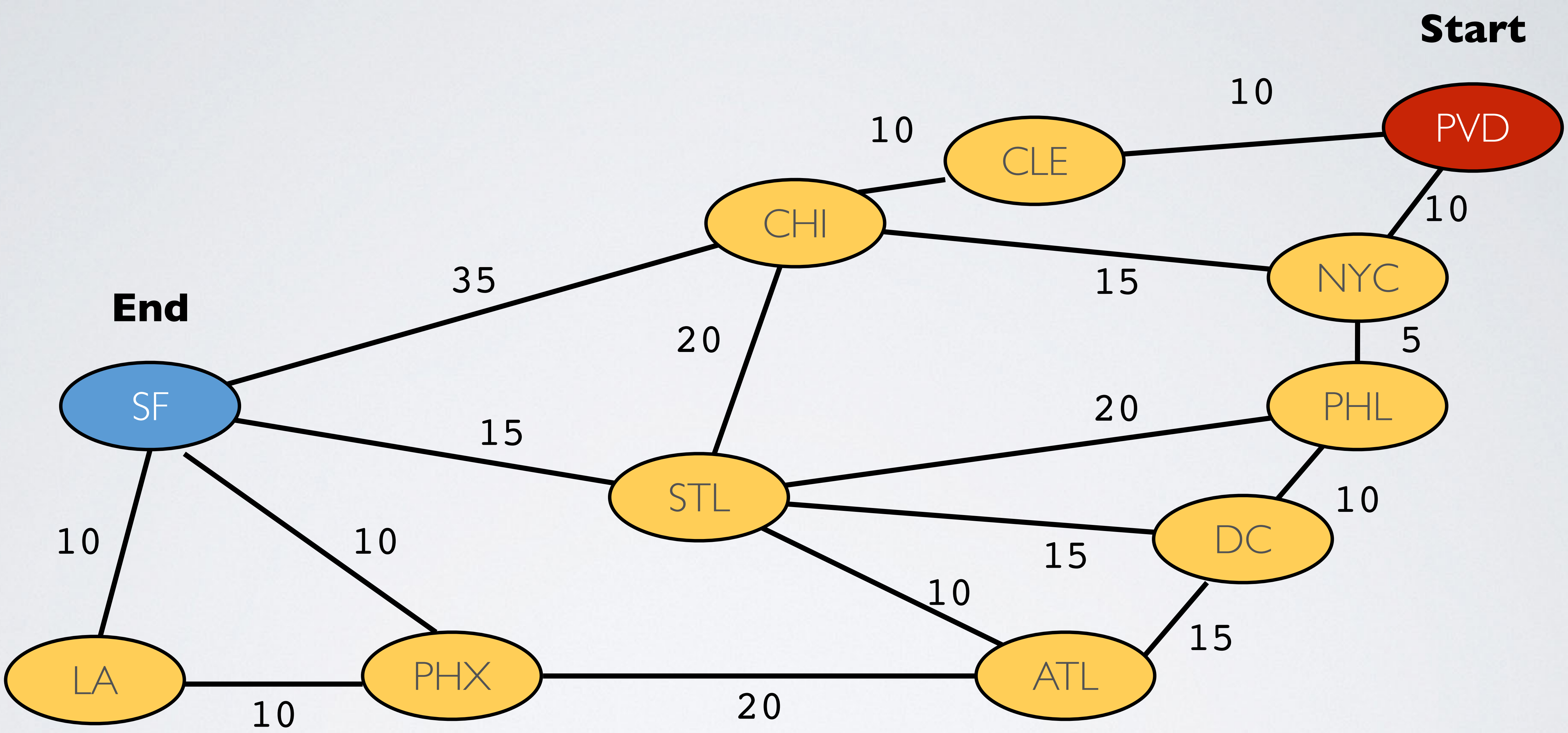
- SSSP problem: find shortest paths to all other nodes in a graph from a particular starting node
- Graph can be directed or undirected (we'll present on undirected graphs)
- *Edges can have weights*

Train trip!



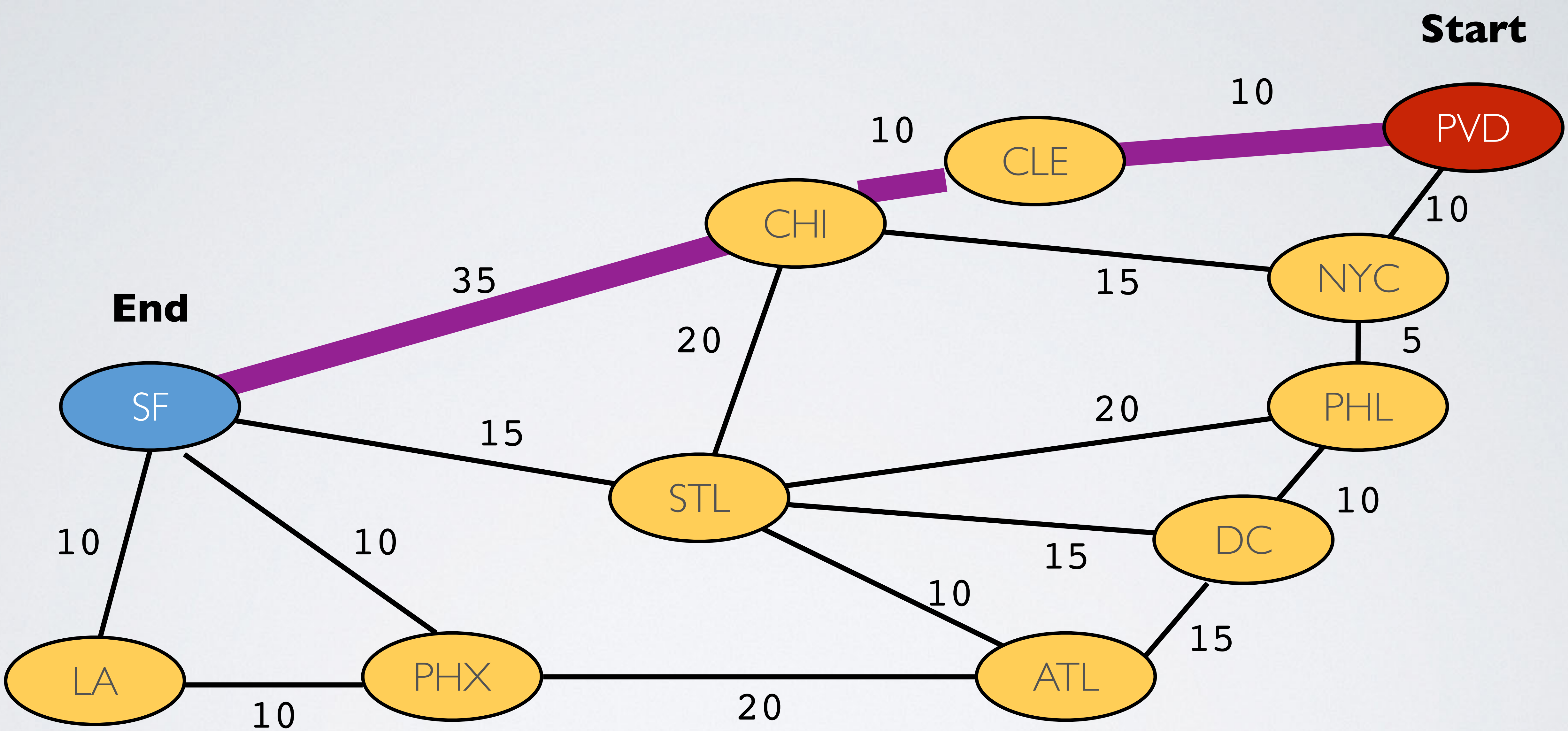
Start
PVD
10
CLE
10
CHI
10
NYC
15
35
End
SF
20
5
PHL
20
15
STL
10
DC
10
10
15
10
15
LA
PHX
ATL
10
20

Train trip!



What's the trip between PVD->SF that makes fewest stops?

Train trip!



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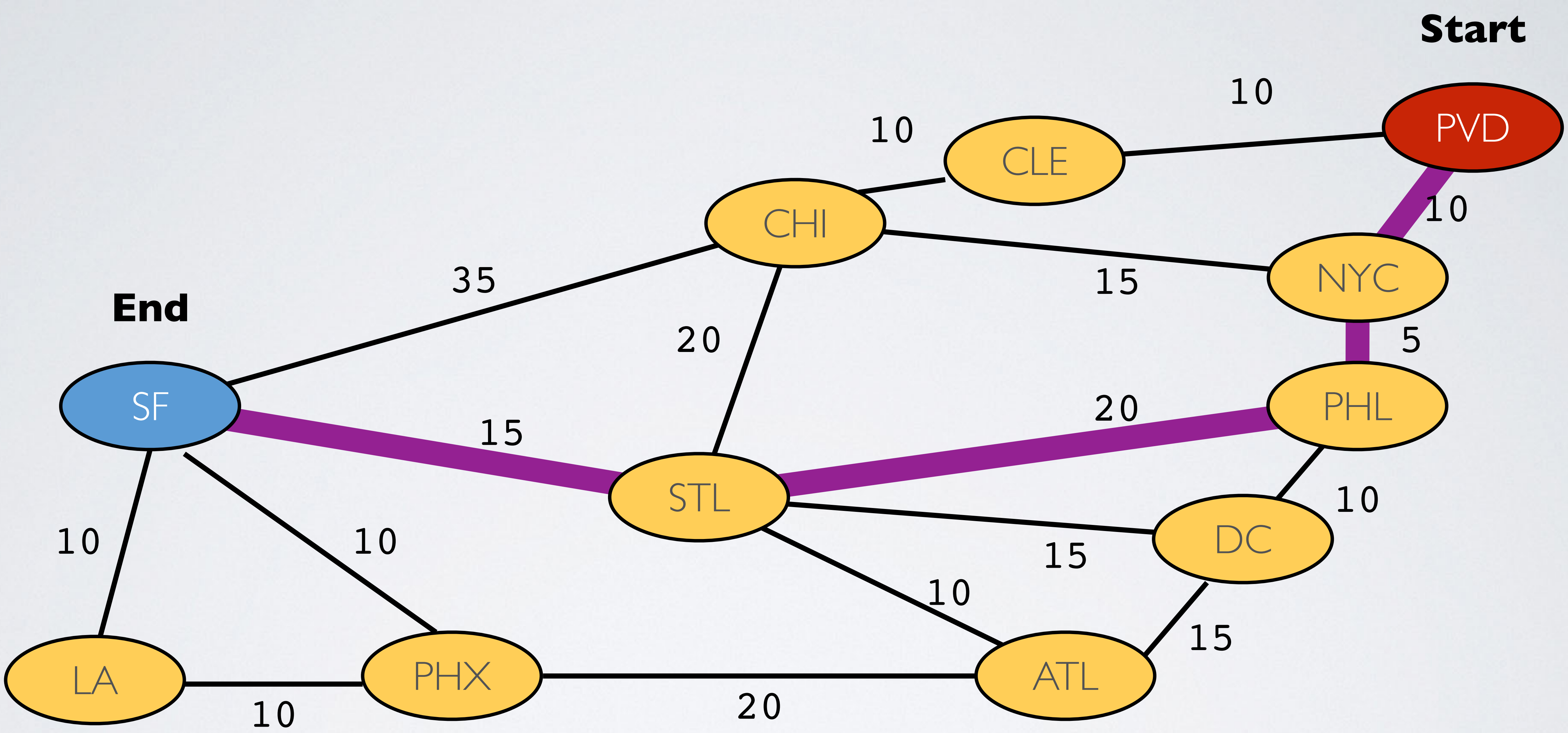
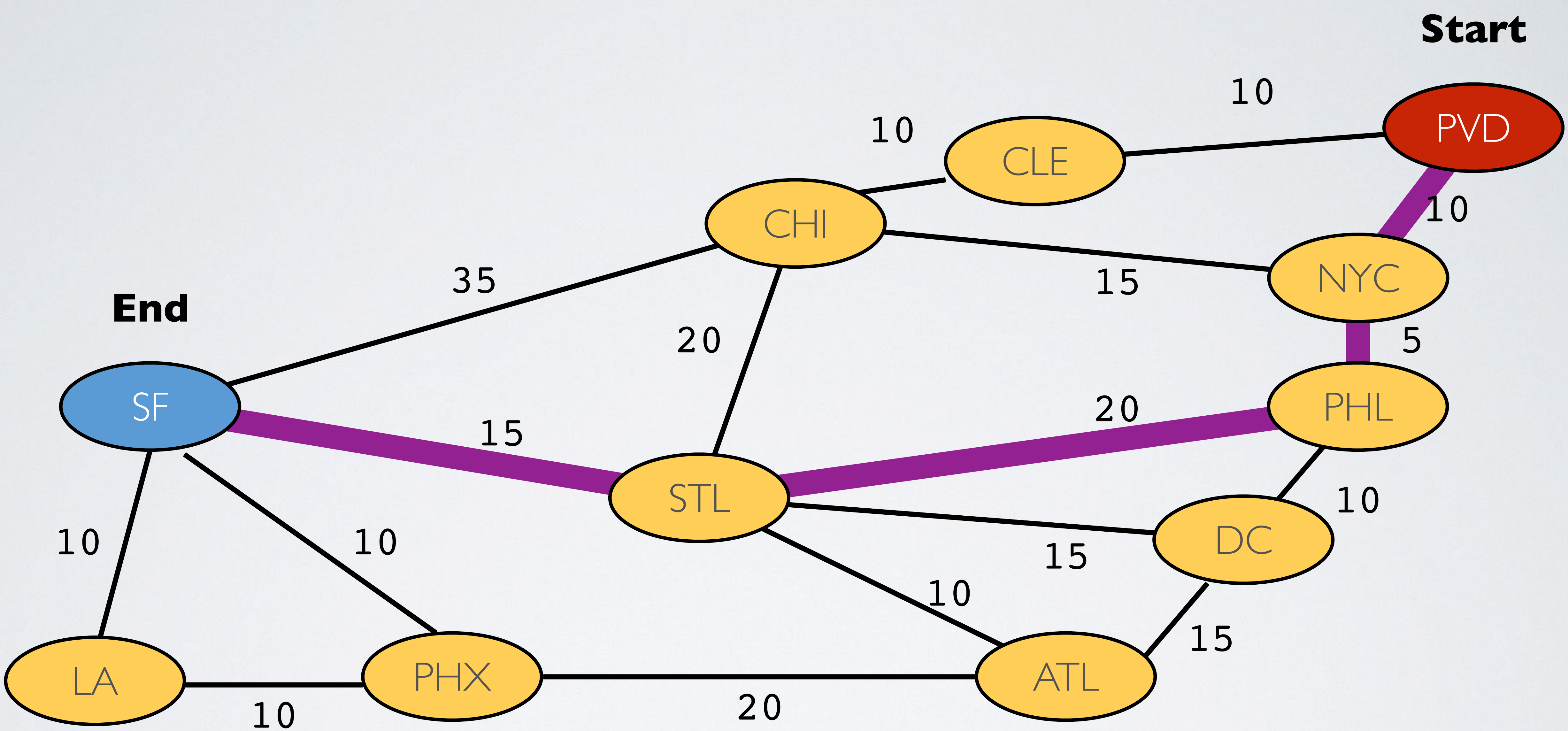


What's the *cheapest* trip?

Train trip!



What's the *cheapest* trip?

BFS ignores edge weights!

Shortest Path

- Why does BFS work with unit edges?
  - Nodes visited *in order of total distance* from source
- We need way to do the same even when edges have distinct weights!

Can we modify BFS?

```
function distance(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest cost of
   //         a path from the source.

   for every node in G:
      node.stops = infinity
      node.previous = null

   Q = new Queue()
   source.stops = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      airport = Q.dequeue()
      for neighbor in airport's adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            neighbor.stops = airport.stops + 1
            neighbor.previous = airport
            Q.enqueue(neighbor)
```

Can we modify BFS?

```
function distance(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest cost of
   //         a path from the source.

   for every node in G:
      node.distance = infinity
      node.previous = null

   Q = new Queue()
   source.distance = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      node = Q.dequeue()
      for neighbor in nodes's adjacent nodes:
         if not neighbor.visited:
            neighbor.visited = true
            neighbor.distance = node.distance + 1
            neighbor.previous = node
            Q.enqueue(neighbor)
```

Can we modify BFS?

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function distance(G, source):
   //Input: G: graph, source: vertex
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   for every node in G:
      node.distance = infinity
      node.previous = null

   Q = new Queue()
   source.distance = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      node = Q.dequeue()
      for neighbor in nodes's adjacent nodes:
         if node.distance + cost(node, neighbor) < neighbor.distance:
            neighbor.visited = true
            neighbor.distance = node.distance + 1
            neighbor.previous = node
            Q.enqueue(neighbor)
```

Can we modify BFS?

```
function distance(G, source):
   //Input: G: graph, source: vertex
   //Output: Nothing
   //Purpose: decorate each vertex with the lowest cost of
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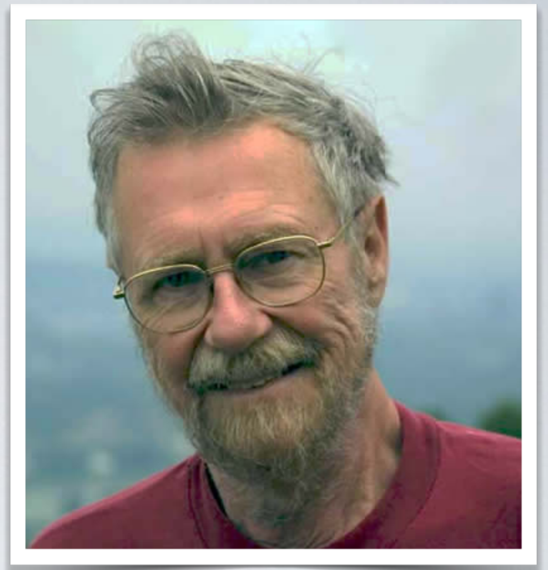
   for every node in G:
      node.distance = infinity
      node.previous = null

   Q = new Queue()
   source.distance = 0
   source.visited = true
   Q.enqueue(source)
   while Q is not empty:
      node = Q.dequeue()
      for neighbor in nodes's adjacent nodes:
         if node.distance + cost(node, neighbor) < neighbor.distance:
            neighbor.distance = node.distance + cost(node, neighbor)
            neighbor.previous = node
            somehow add neighbor to Q at the right place
```

Shortest Path

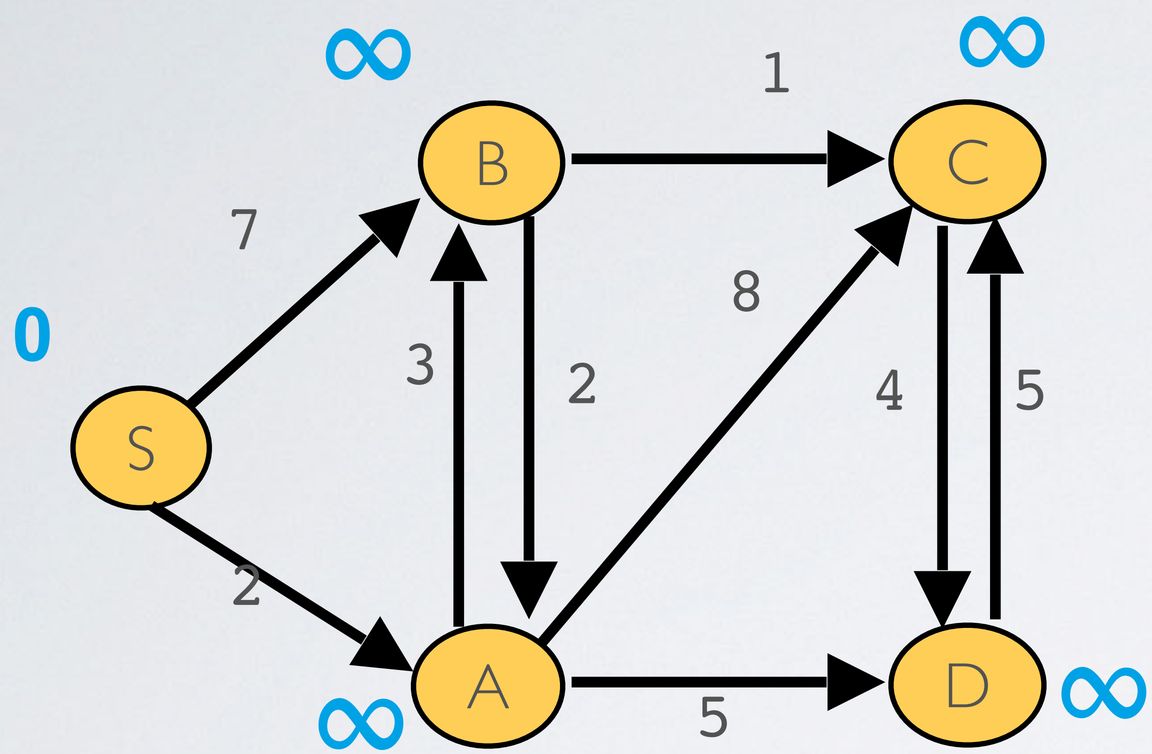
- Use a priority queue!
  - where priorities are total distances from source
  - By visiting nodes in order returned by **`removeMin()`**…
  - …you visit nodes in order of how far they are from source
- You guarantee shortest path to node because…
  - …you don't explore a node until all nodes closer to source have already been explored

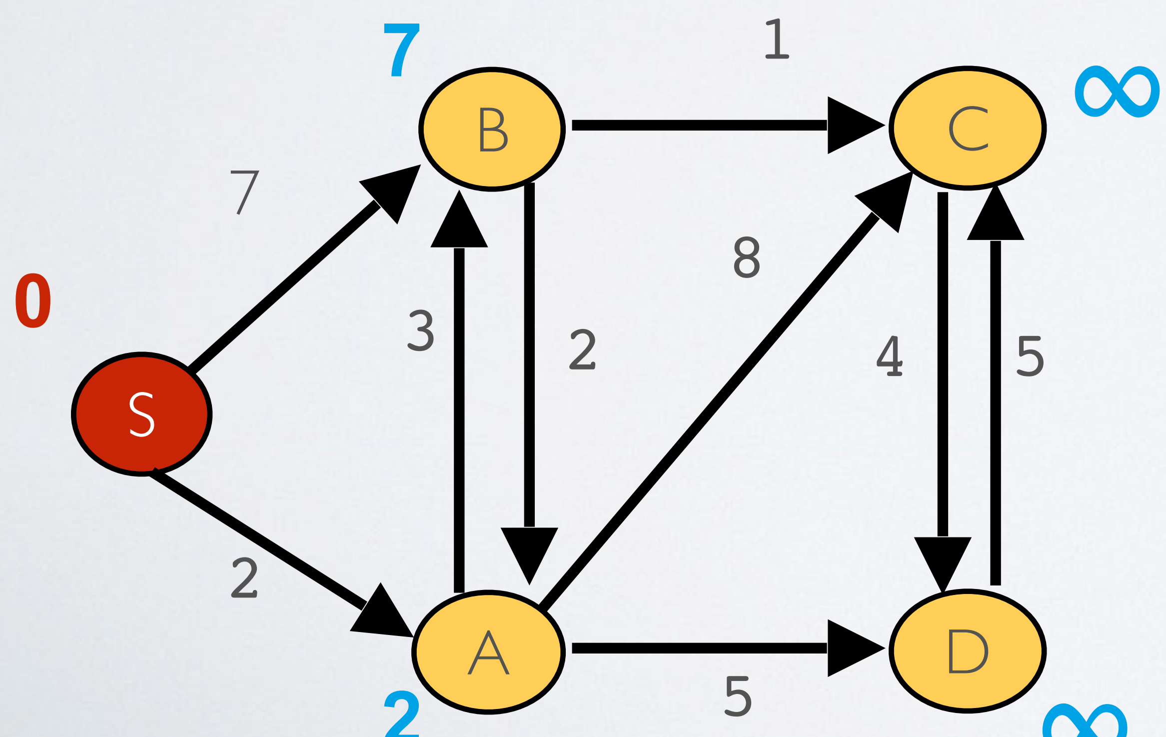
Dijkstra’s Algorithm



- The algorithm is as follows:
  - Decorate source with distance 0 & all other nodes with ∞
  - Add all nodes to priority queue w/ distance as priority
  - While the priority queue isn’t empty
    - Remove node from queue with minimal priority
    - Update distances of the removed node’s neighbors if distances decreased
- When algorithm terminates, every node is decorated with minimal cost from source

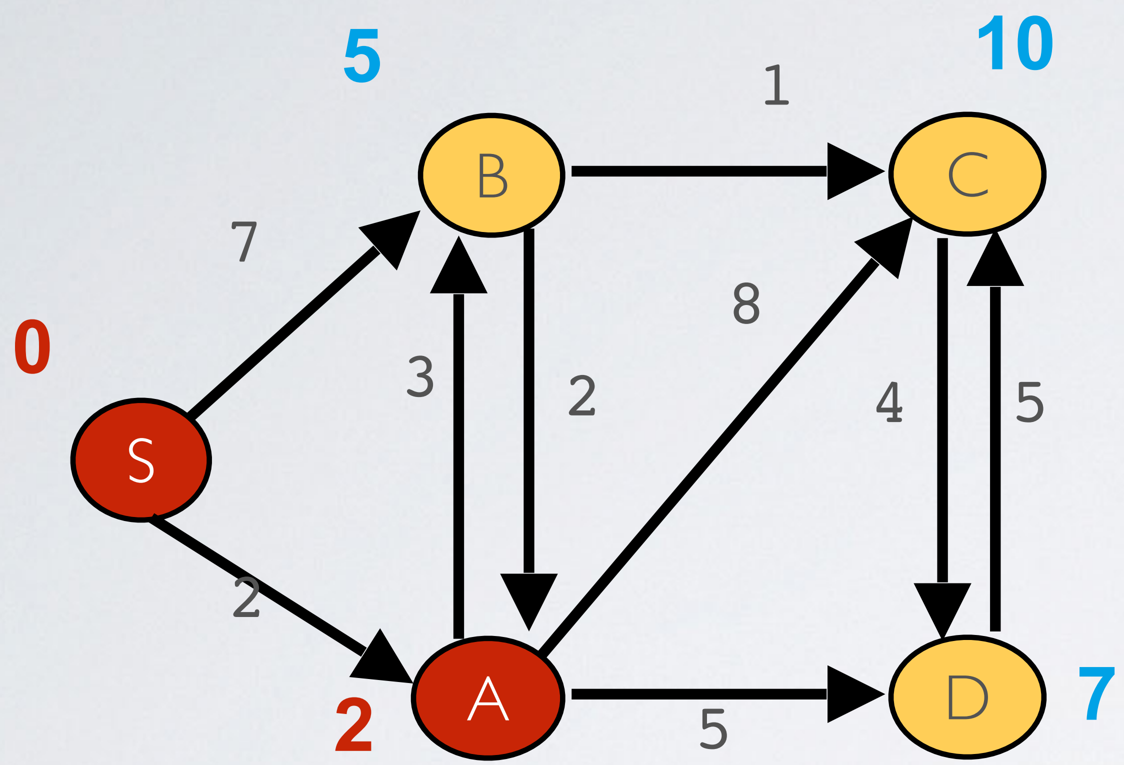
Dijkstra's Algorithm Example

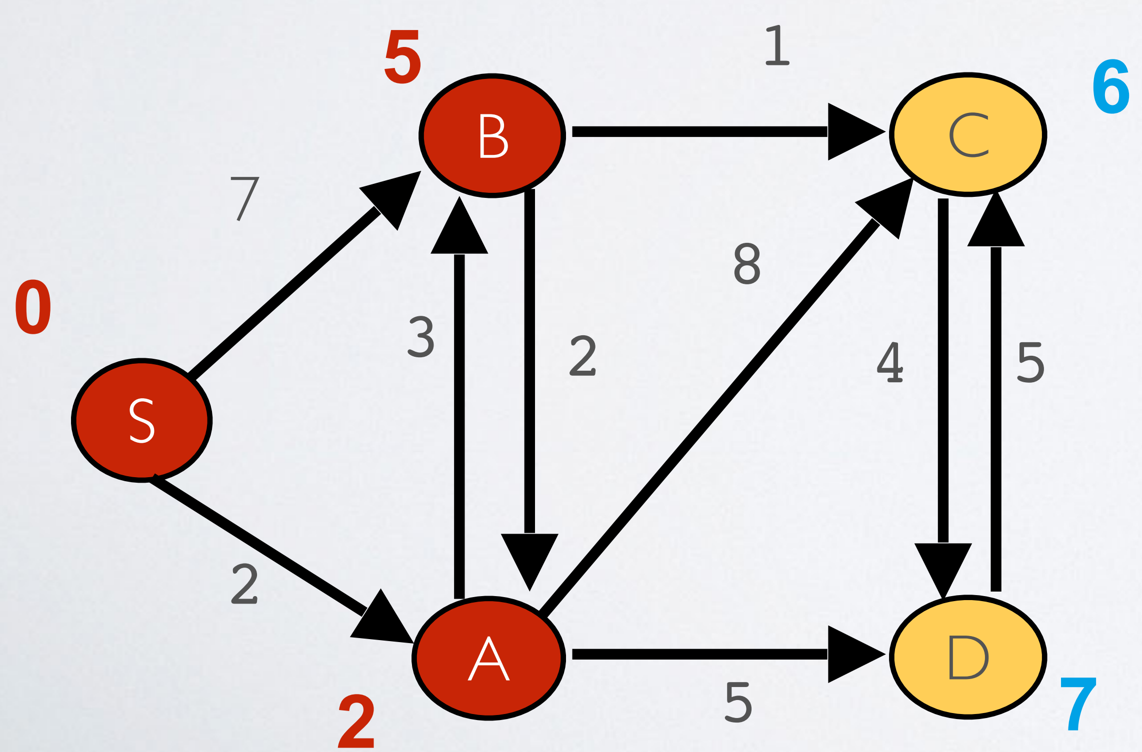




- Step 1
  - Label source w/ dist. 0
  - Label other vertices w/ dist. ∞
  - Add all nodes to `Q`
- Step 2
  - Remove node with min. priority from `Q` (`S` in this example).
  - Calculate dist. from source to removed node's neighbors…
  - …by adding adjacent edge weights to `S`'s dist.

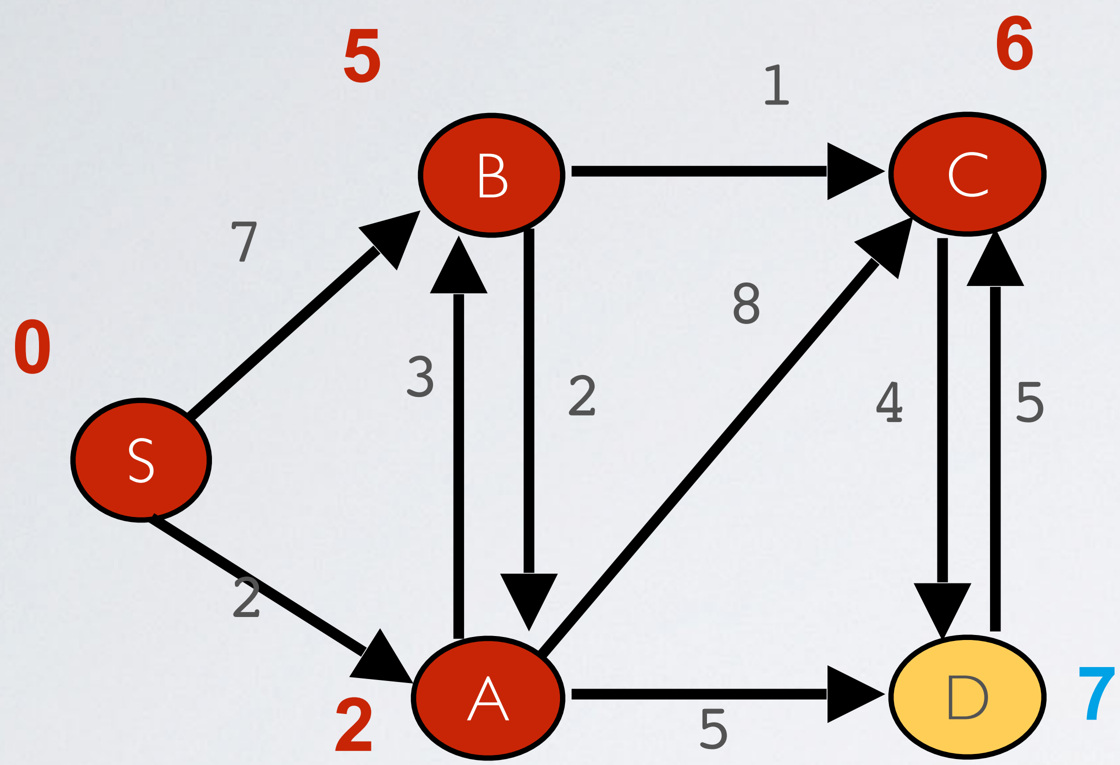
Dijkstra's Algorithm Example

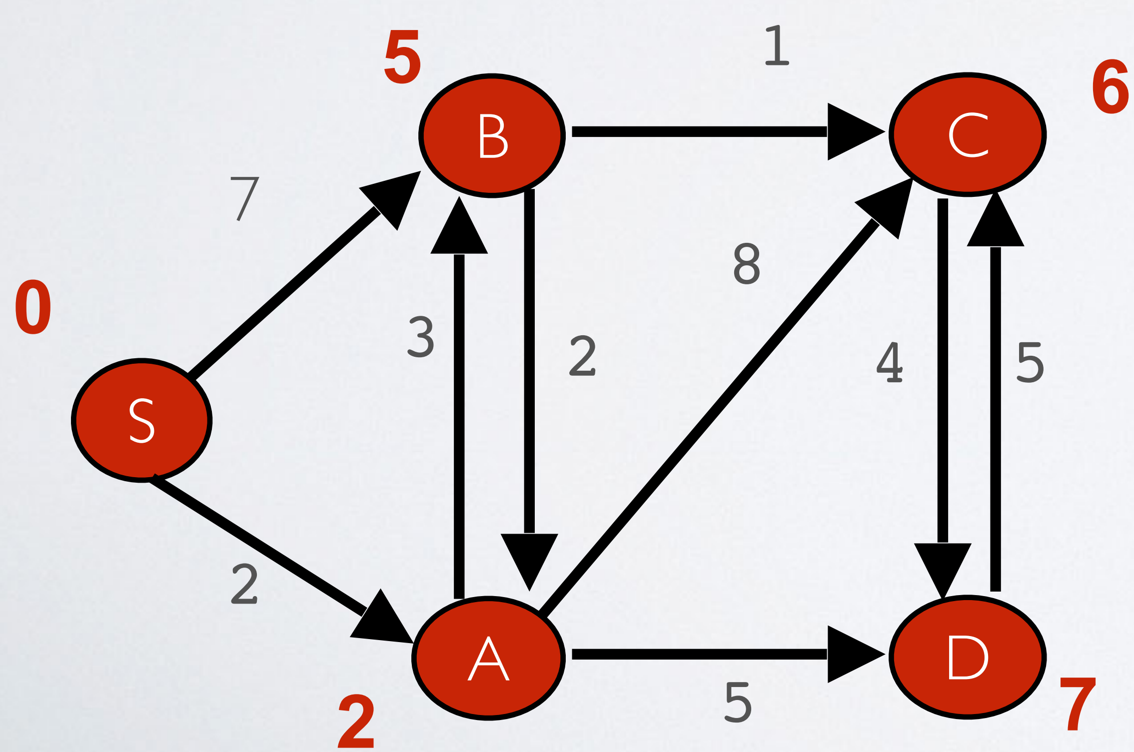




- Step 3
  - While `Q` isn't empty,
    - repeat previous step
    - removing `A` this time
  - Priorities of nodes in `Q` may have to be updated
    - ex: `B`'s priority
- Step 4
  - Repeat again by removing vertex `B`
  - Update distances that are shorter using this path than before
  - ex: C now has a distance `6` not `10`

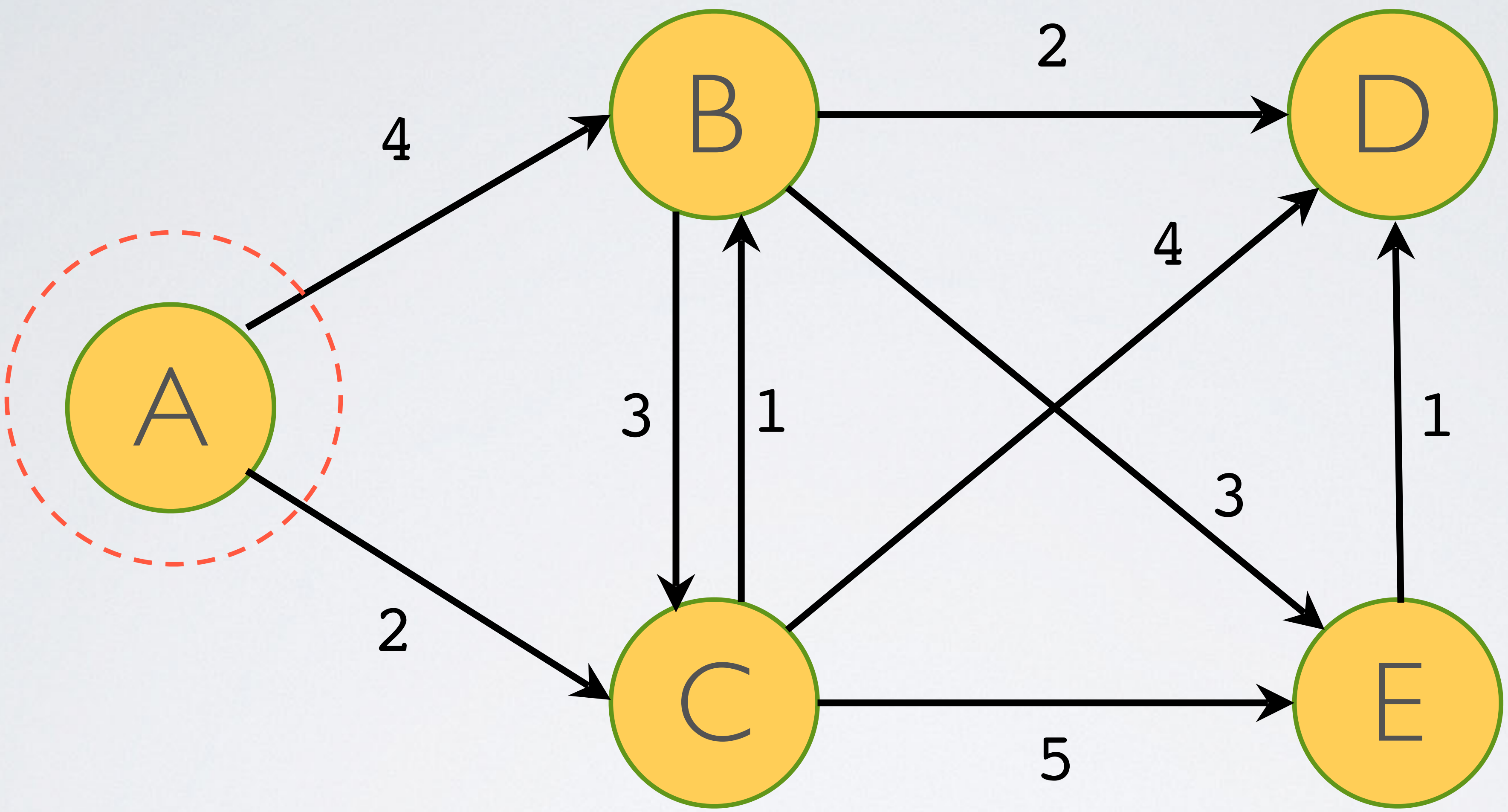
Dijkstra’s Algorithm Example





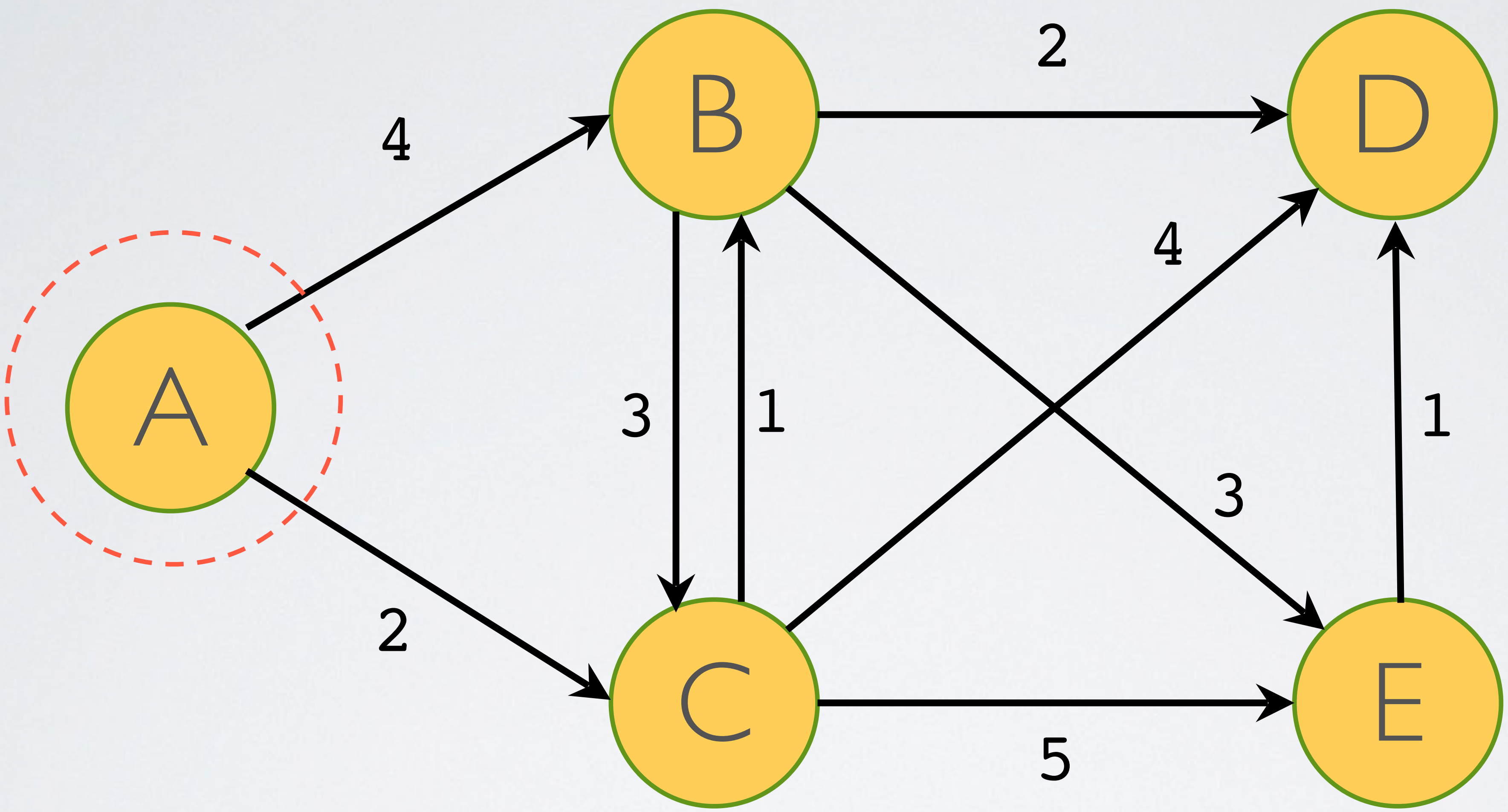
- Step 5
  - Repeat
    - this time removing C
- Step 6
  - After removing D…
  - …every node has been visited…
  - …and decorated w/ shortest dist. to source

Dijkstra's Example 2



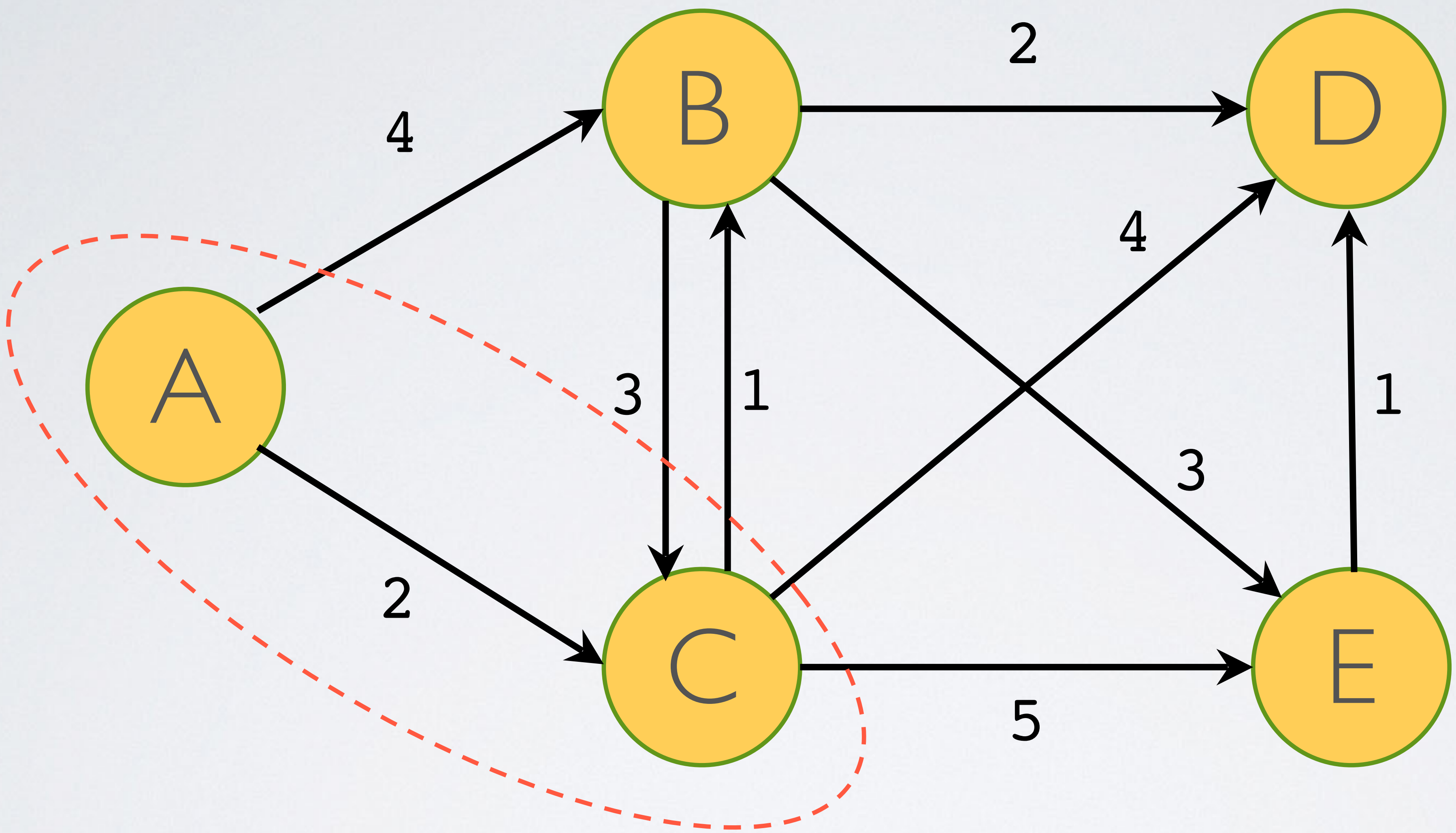
A	B	C	D	E
0	∞	∞	∞	∞

Dijkstra's Example



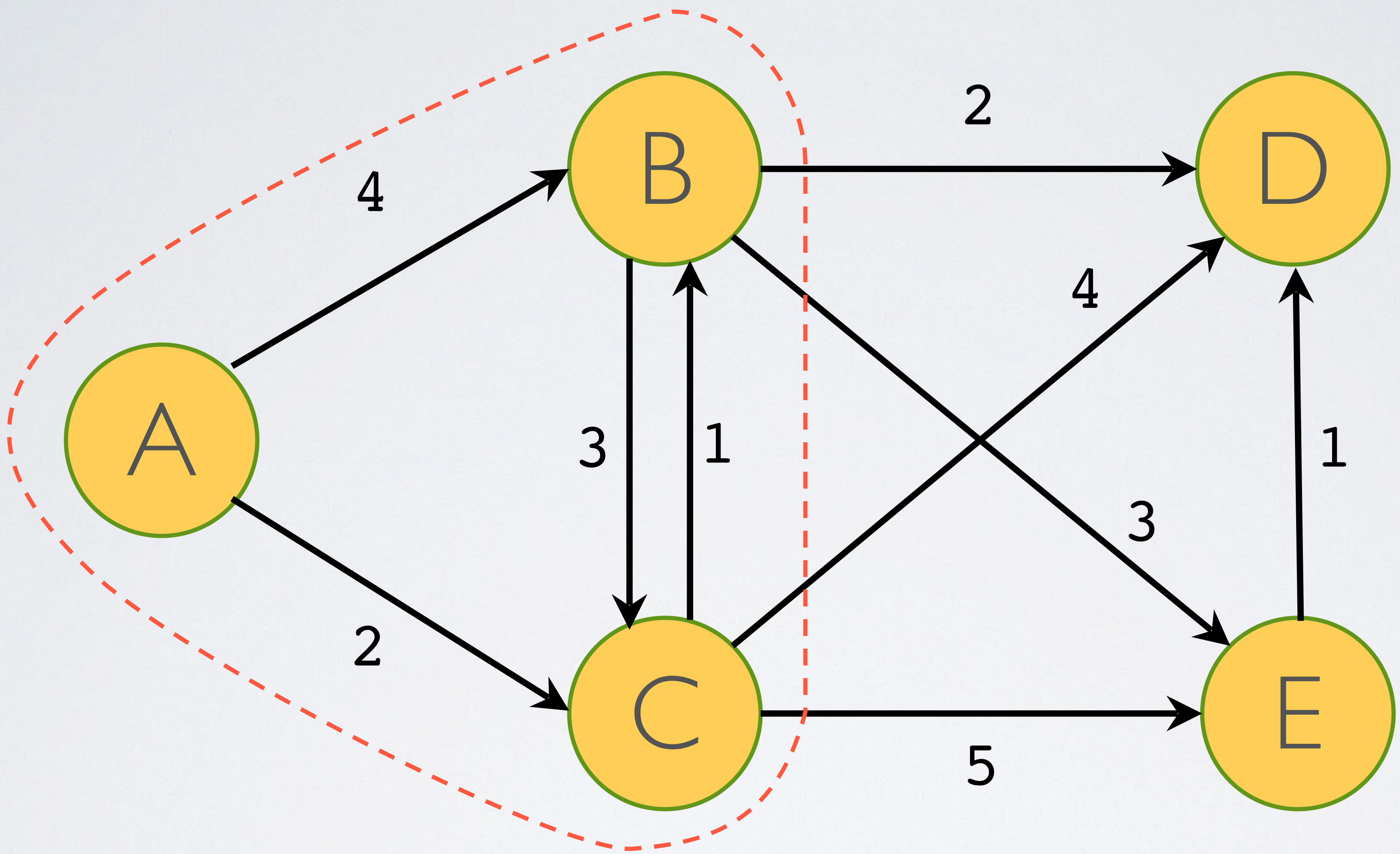
A	B	C	D	E
0	4	2	∞	∞

Dijkstra's Example



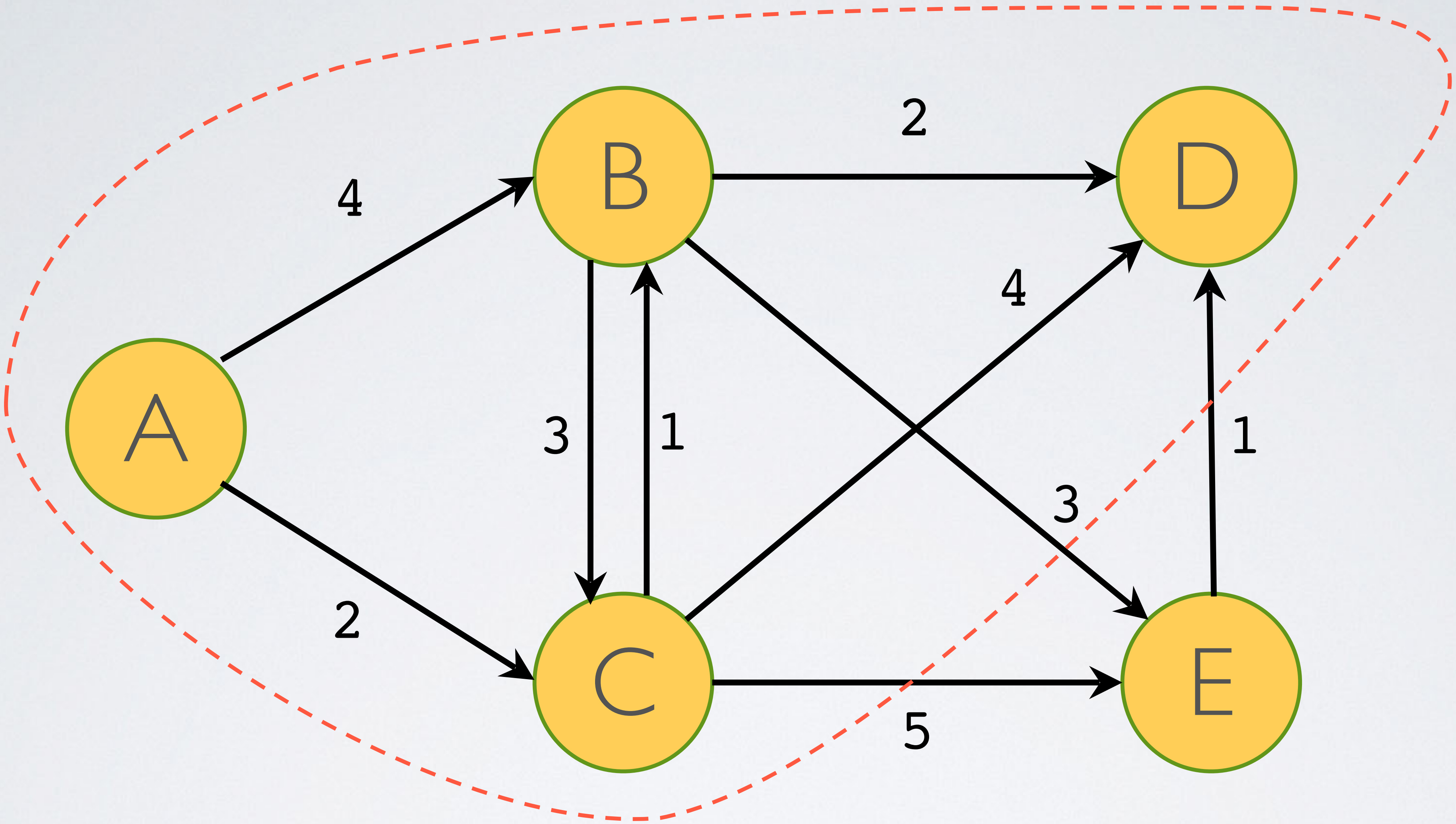
A	B	C	D	E
0	**3**	2	**6**	**7**

Dijkstra's Example



A	B	C	D	E
0	3	2	**5**	**6**

Dijkstra's Example



A	B	C	D	E
0	3	2	5	6

Dijkstra's Algorithm

- Comes up with an *optimal* solution
  - *shortest* path to each node
- Like many optimization algorithms, uses dynamic programming
  - overlapping subproblems (distances to nodes)
  - solved in a particular order (closest first)
- Dijkstra's is *greedy*
  - at each step, considers next closest node
  - Greedy algorithms not always optimal, usually fast

Dijkstra Pseudo-Code

```
function dijkstra(G, s):
   // Input: graph G with vertices V, and source s
   // Output: Nothing
   // Purpose: Decorate nodes with shortest distance from s
   for v in V:
     v.dist = infinity  // Initialize distance decorations
     v.prev = null      // Initialize previous pointers to null
   s.dist = 0          // Set distance to start to 0

   PQ = PriorityQueue(V)    // Use v.dist as priorities
   while PQ not empty:
      u = PQ.removeMin()
      for all edges (u, v):  //each edge coming out of u
         if u.dist + cost(u, v) < v.dist: // cost() is weight
            v.dist = u.dist + cost(u,v)   // Replace as necessary
            v.prev = u       // Maintain pointers for path
            PQ.decreaseKey(v, v.dist)
```

Dijkstra Runtime w/ Heap

- If PQ implemented with Heap
  - `insert( )` is `O(log|V|)`
    - you may need to upheap
  - `removeMin( )` is `O(log|V|)`
    - you may need to downheap
  - `decreaseKey()` is `O(log|V|)`
    - assume we have dictionary that maps vertex to heap entry in `O(log|V|)` time (so no need to scan heap to find entry)
    - you may need to upheap after decreasing the key

Dijkstra Runtime w/ Heap

```
function dijkstra(G, s):
    for v in V:                          <- O(|V|)
        v.dist = infinity
        v.prev = null
    s.dist = 0

    PQ = PriorityQueue(V)                <- O(|V|log|V|)
    while PQ not empty:                  <- O(|V|)
        u = PQ.removeMin()               <- O(log|V|)
        for all edges (u, v):            <- O(|E|) total
            if v.dist > u.dist + cost(u, v):
                v.dist = u.dist + cost(u,v)
                v.prev = u
                PQ.decreaseKey(v, v.dist)    <- O(log|V|)
```

Dijkstra Runtime w/ Heap

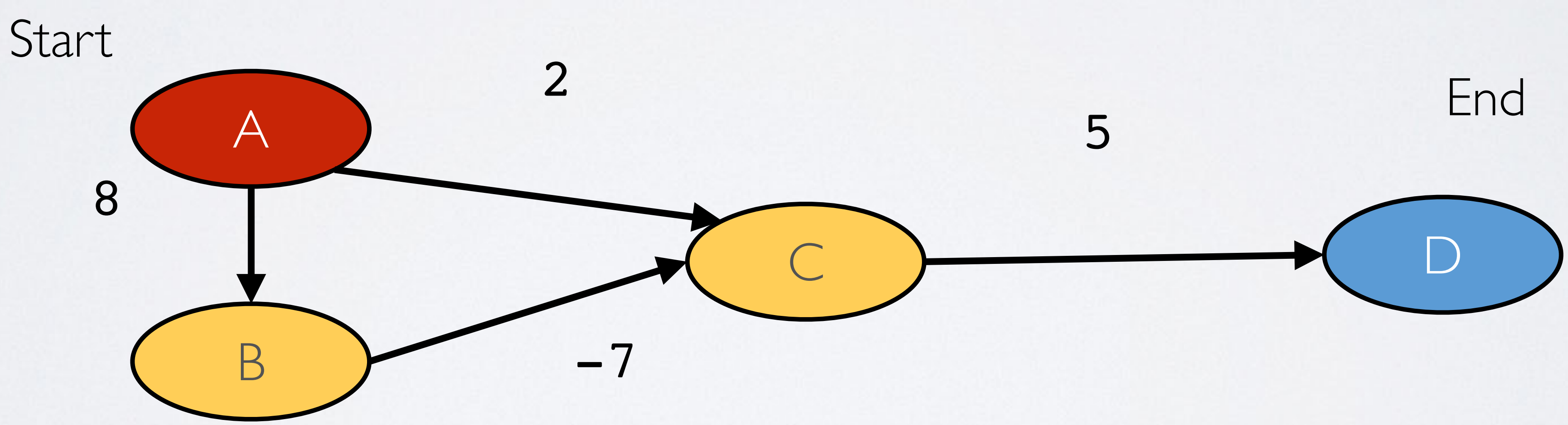
- If PQ implemented with Heap

$$
\begin{aligned}
&O(|V| + |V| \log |V| + |V| \log |V| + |E| \log |V|) \\
&\qquad = O(|V| + |V| \log |V| + |E| \log |V|) \\
&\qquad = O\Big((|V| + |E|) \cdot \log |V| \Big)
\end{aligned}
$$

- Note
  - though the `O(|E|)` loop is nested in the `O(|V|)` loop
  - we visit each edge at most twice rather than `|V|` times
  - That's why while loop is $O\Big((V \log |V|) + (|E| \log |V|) \Big)$

Dijkstra isn't perfect!

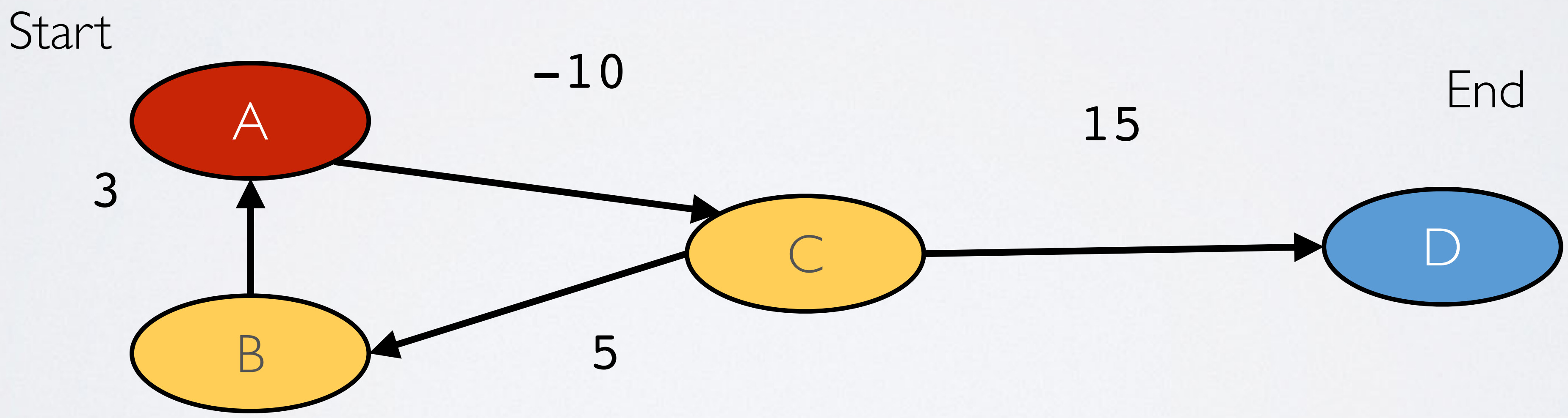
- We can find shortest path on weighted graph in
  - `O((|V|+|E|)×log|V|)`
  - or can we…
- Dijkstra fails with negative edge weights



- Returns **`[A,C,D]`** when it should return **`[A,B,C,D]`**

Negative Edge Weights

- Negative edge weights are problem for Dijkstra
- But negative cycles are even worse!
  - because there is no true shortest path!



Bellman-Ford Algorithm

- Algorithm that handles graphs w/ neg. edge weights
- Similar to Dijkstra's but more robust
  - Returns same output as Dijkstra's for any graph w/ only positive edge weights (but runs slower)
  - Returns correct shortest paths for graphs w/ neg. edge weights
  - How: not greedy!