Tree Properties & Traversals

CS16: Introduction to Data Structures & Algorithms
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Outline

- Tree & Binary Tree ADT
- Tree Traversals
  - Breadth-First Traversal
  - Depth-First Traversal
- Recursive DFT
  - pre-order, post-order, in-order
- Euler Tour Traversal
- Traversal Problems
- Analysis on perfect binary trees
What is a Tree?

- Abstraction of hierarchy
- Tree consists of
  - nodes with parent/child relationship
- Examples
  - Files/folders (Windows, MacOSX, …, CS33)
  - Merkle Trees (Bitcoin, CS166)
  - Encrypted Dictionaries, Oblivious RAMs (CS2950-v)
  - Datacenter Networks (Azure, AWS, Google, CS168)
  - Distributed Systems (Distributed Storage, Cluster computing, CS138)
  - AI & Machine Learning (Decision trees, Neural Networks, CS141, CS142)
Tree “Anatomy”

internal nodes

subtree

leaves/external nodes

d dotted lines:

root

height
Tree Terminology

- **Root**: node without a parent (A)
- **Internal node**: node with at least one child (A, B, C, F)
- **External node (leaf)**: node without children (E, I, J, K, G, H, D)
- **Parent node**: node immediately above a given node (parent of C is A)
- **Child node**: node(s) immediately below a given node (children of C are G and H)
- **Ancestors of a node**:
  - parent, grandparent, grand-grandparent, etc. (ancestors of G are C, A)
- **Depth of a node**: number of ancestors (I has depth 3)
- **Height of a tree**:
  - maximum depth of any node (tree with just a root has height 0, this tree has height 3)
- **Descendant of a node**: child, grandchild, grand-grandchild, etc.
- **Subtree**: tree consisting of a node and its descendants
Tree ADT

- Tree methods:
  - int size(): returns the number of nodes
  - boolean isEmpty(): returns true if the tree is empty
  - Node root(): returns the root of the tree

- Node methods:
  - Node parent(): returns the parent of the node
  - Node[] children(): returns the children of the node
  - boolean isInternal(): returns true if the node has children
  - boolean isExternal(): returns true if the node is a leaf
  - boolean isRoot(): returns true if the node is the root
Binary Trees

- Internal nodes have at most 2 children
  - left and right
  - if only 1 child, still need to specify if left or right
- Recursive definition
  - Tree with a single node or
  - Tree whose root has at most 2 children
    - each of which is a binary tree
Binary Tree ADT

- In addition to Tree methods binary trees have:
  - Node `left()`: returns the left child if it exists, else NULL
  - Node `right()`: returns the right child if it exists, else NULL
  - Node `hasLeft()`: returns TRUE if node has left child
  - Node `hasRight()`: returns TRUE if node has right child
Perfection

- A binary tree is a **perfect** if
  - every level is completely full

![Not perfect](image1)

![Perfect!](image2)
Completeness

- A binary tree is **left-complete** if
  - every level is completely full, possibly excluding the lowest level
  - all nodes are as far left as possible
Aside: Decorations

- Decorating a node
  - associating a value to it

- Two approaches
  - Add new attribute to each node
    - ex: `node.numDescendants = 5`
  - Maintain dictionary that maps nodes to decoration
    - do this if you can’t modify tree
    - ex: `descendantDict[node] = 5`
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Tree Traversals

- Not obvious how to visit every node of a tree
  - unlike arrays which have a natural order
- Tree traversal
  - algorithm for visiting every node in a tree
- Many possible tree traversals
  - each visit nodes in different orders
Breadth- vs. Depth-First Traversals

Diagram showing two trees:
- Breadth-first traversal:
  - Node A
  - Node B
  - Node C
  - Node D
  - Node E
  - Node F
  - Node G
  - Node H
  - Node I
- Depth-first traversal:
  - Node A
  - Node B
  - Node C
  - Node D
  - Node E
  - Node F
  - Node G
  - Node H
  - Node I
Traversal Strategy

- Array traversal can be done with a **for** loop
  - Why?

- What about a tree traversal?
  - Why not?

- For tree traversal we’ll use a **while** loop
function traversal(root):
    ...
    while ...
    ...
    do something with a node
    ...

function traversal(root):
    ...
    while ...
    ...
    do something with node
    store children in S

function traversal(root):
    ...
    while S is not empty
    ...
    do something with node
    store children in S
function `traversal(root):`

...  
while S is not empty
  ...
  do something with node
  store children in S

function `traversal(root):`

...  
while S is not empty
  get node from S
  do something with node
  store children in S

function `traversal(root):`

  Store root in S
  while S is not empty
    get node from S
    do something with node
    store children in S
Traversal Strategy

function **traversal**(root):
    Store root in S
    while S is not empty
        get node from S
        do something with node
        store children in S

- What is S exactly?
  - Where we store nodes until we can process them
- Which node of S should we process next?
  - the first? the last?
function `traversal(root)`:
  Store root in S
  while S is not empty
    get node from S
    do something with node
    store children in S
function traversal(root):
    Store root in S
    while S is not empty
        get node from S
        do something with node
        store children in S

Does S remind you of something?
Traversals Strategy — Get First

- If we get first node in S
  - we’re doing FIFO…
  - so S is a queue!
  - Traversal w/ Queue gives breadth-first traversal

- Why?
  - Queue guarantees node is processed before its children
  - Children can be inserted in any order

```python
function bft(root):
    Q = new Queue()
    enqueue root
    while Q is not empty
        node = Q.dequeue()
        visit(node)
        enqueue node’s left & right children
```
Breadth-First Traversal

- Start at root
  - Visit both of its children first,
    - Then all of its grandchildren,
      - Then great-grandchildren
        - etc…
- Also known as
  - level-order traversal
Depth-First Traversal

- What if we process youngest/last element in S?
  - we’re doing LIFO…
  - so S is a stack!
  - Traversal w/ Stack gives us…
- Depth-first search
  - start from root
  - traverse each branch before backtracking
  - can produce different orders
Depth-First Traversal

```python
function dft(root):
    S = new Stack()
push root
while S is not empty
    node = S.pop()
    visit(node)
push node’s left & right children
```

- Why does Stack give DFT?
  - Stack guarantees entire branch will be visited before visiting another branch
- Children can be pushed on stack in any order
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Recursive Depth-First Traversal

- DFT can be implemented recursively
- With recursion we can have 3 different orders
  - **pre-order**: visits each node before visiting left and right
  - **post-order**: visits each child before visiting node
  - **in-order**: visits left child, node and then right child
Depth-First Visualizations
Pre-order Traversal

function preorder(node):
    visit(node)
    if node has left child
        preorder(node.left)
    if node has right child
        preorder(node.right)

Note: like iterative DFT
function **postorder**(node):
  if node has left child
    postorder(node.left)
  if node has right child
    postorder(node.right)
  visit(node)
In-order Traversal

**Function** `inorder(node)`:
- If node has left child, `inorder(node.left)`
- Visit(node)
- If node has right child, `inorder(node.right)`
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When to Use What Traversal?

- How do you know which traversal to use?
- Sometimes it doesn’t matter
- Often one traversal makes solving problem easier
Tree Traversals Problems

Which traversal should be used to decorate nodes with # of descendants?

Activity #1
Which traversal should be used to decorate nodes with # of descendants?
Which traversal should be used to decorate nodes with # of descendants?
Tree Traversals Problems

- Decorating with number of descendants?
- **Post-order**
  - visits both children before node
  - easy to calculate # of descendants if you know # of descendants of both children
  - try writing pseudo-code for this
Given root, which traversal should be used to test if tree is perfect?
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Tree Traversals Problems

- Decorating with number of descendants?
- **Breadth-first**
  - traverses tree level by level
  - keep track of how many nodes at level
  - each level should have twice as many as previous level
  - There are other ways to solve this. Can you think of any?
Tree Traversals Problems

- Evaluate arithmetic expression tree

$$(7 - (4 + 3)) + (9 / 3) =$$

- Best traversal?
  - **post-order**: to evaluate operation, you first need to evaluate sub-expression on each side
  - What should you do when you get to a leaf?
Tree Traversals Problems

- Given tree, print out expression w/o parentheses

```
7 - 4 + 3 + 9 / 3
```

- Best traversal?

  - **in-order**: gives nodes from left to right
Euler Tour Traversal

- Generic traversal of binary tree
  - pre-order, post-order and in-order are special cases
- Each node visited 3 times
  - left, bottom, right
Euler Tour Traversal

- Visit on
  - **left:** pre-order traversal
  - **bottom:** in-order traversal
  - **right:** post-order traversal
Euler Tour Traversal

function eulerTour(node):
  # pre-order
  visitLeft(node)

  if node has left child:
    eulerTour(node.left)

  # in-order
  visitBelow(node)

  if node has right child:
    eulerTour(node.right)

  # post-order
  visitRight(node)
Tree Traversals Problems

- Given tree, print out expression w/ parentheses

\[ (7 - (4 + 3)) + (9 / 3) \]

- Best traversal?

- Euler tour
Tree Traversals Problems

- Best traversal?
  - **Euler tour**
- Internal nodes
  - For pre-order visit, print "("
  - For in-order visit, print operator
  - For post-order visit, print ")"
- Leaves
  - Don’t do anything for pre-order and post-order visits
  - For in-order visit, print number
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Analyzing Binary Trees

- Many things can be modeled as binary trees
  - ex: Fibonacci recursive tree
- Knowing certain facts about binary trees helps with analysis
  - ex: how many recursive calls are made by a binary recursive tree of height n?
- Perfect binary trees are easier to analyze…
- …so we use them to estimate analysis of general trees
Analyzing **Perfect** Binary Trees

- Number of nodes PBT of height $h$:
  - $2^{h+1} - 1$

- Height of a PBT with $n$ nodes:
  - $\log_2(n+1)$

- Number of leaves in PBT of height $h$:
  - $2^h$

- Number of nodes in PBT with $L$ leaves:
  - $2L - 1$
Induction on Perfect Binary Trees

- Can use induction to prove things about PBTs
- Alternative definition of perfect binary trees

Tree T is a perfect binary tree if

- it has only one node
- or has root with left and right subtrees which are both perfect binary trees of same height
- (if subtrees have height $h$, then $T$ has height $h+1$)
Example Inductive Proof on PBTs

- Prove $\mathbf{P}(n)$:
  - number of nodes in a PBT of height $n$ is $2^{n+1} - 1$

- Base case $\mathbf{P}(0)$:
  - $2^{0+1} - 1 = 2 - 1 = 1$
  - Number of nodes in PBT of height 0 is 1, because tree only has a root by definition

- Inductive hypothesis:
  - assume $\mathbf{P}(k)$ is true (for some $k \geq 0$)
  - in other words: the number of nodes in PBT of height $k$ is $2^{k+1} - 1$
Then prove that $P(k+1)$ is true:

- Let $T$ be any PBT of height $k+1$

- By definition, $T$ consists of root with two subtrees, $L$ and $R$, which are both PBTs of height $k$

- By inductive hypothesis, $L$ and $R$ both have $2^{k+1} - 1$ nodes

- Number of nodes in $T$ is therefore:
  \[1 + 2 \times (2^{k+1} - 1) = 1 + 2^{k+2} - 2 = 2^{(k+1)+1} - 1\]

- Since we've proved
  
  - $P(0)$ is true
  - $P(k)$ implies $P(k+1)$ for any $k \geq 0$

  It follows by induction that $P(n)$ is true for all $n \geq 0$