Tree Properties & Traversals

CS16: Introduction to Data Structures & Algorithms
Spring 2019
How does OS calculate size of directories?
Outline

- Tree & Binary Tree ADT
- Tree Traversals
  - Breadth-First Traversal
  - Depth-First Traversal
- Recursive DFT
  - pre-order, post-order, in-order
- Euler Tour Traversal
- Traversal Problems
- Analysis on perfect binary trees
What is a Tree?

- Abstraction of hierarchy
- Tree consists of
  - nodes with parent/child relationship
- Examples
  - Files/folders (Windows, MacOSX, …, CS33)
  - Merkle Trees (Bitcoin, CS166)
  - Encrypted Data Structures (CS2950-v)
  - Datacenter Networks (Azure, AWS, Google, CS168)
  - Distributed Systems (Distributed Storage, Cluster computing, CS138)
  - AI & Machine Learning (Decision trees, CS141, CS142)
Tree “Anatomy”

- **root**
- **internal nodes**
- **subtree**
- **leaves/external nodes**
- **height**
Tree Terminology

- **Root:** node without a parent (A)
- **Internal node:** node with at least one child (A, B, C, F)
- **Leaf (external node):** node without children (E, I, J, K, G, H, D)
- **Parent node:** node immediately above a given node (parent of C is A)
- **Child node:** node(s) immediately below a given node (children of C are G and H)
- **Ancestors of a node:**
  - parent, grandparent, grand-grandparent, etc. (ancestors of G are C, A)
- **Descendant of a node:** child, grandchild, grand-grandchild, etc.
- **Depth of a node:** number of ancestors (I has depth 3)
- **Height of a tree:**
  - maximum depth of any node (tree with just a root has height 0, this tree has height 3)
- **Subtree:** tree consisting of a node and its descendants
Tree ADT

- Tree methods:
  - int `size()`: returns the number of nodes
  - boolean `isEmpty()`: returns true if the tree is empty
  - Node `root()`: returns the root of the tree

- Node methods:
  - Node `parent()`: returns the parent of the node
  - Node[ ] `children()`: returns the children of the node
  - boolean `isInternal()`: returns true if the node has children
  - boolean `isExternal()`: returns true if the node is a leaf
  - boolean `isRoot()`: returns true if the node is the root
Binary Trees

- Internal nodes have at most 2 children
  - left and right
  - if only 1 child, still need to specify if left or right

- Recursive definition of a Binary Tree
  - a single node
  - or a root node with at most 2 children
    - each of which is a binary tree
Binary Tree ADT

- In addition to Tree methods *binary* trees have:
  - Node **left**(): returns the left child if it exists, else NULL
  - Node **right**(): returns the right child if it exists, else NULL
  - Node **hasLeft**(): returns TRUE if node has left child
  - Node **hasRight**(): returns TRUE if node has right child
Perfection

- A binary tree is **perfect** if
  - every level is completely full
Completeness

- A binary tree is **left-complete** if
  - every level is completely full, possibly excluding the lowest level
  - all nodes are as far left as possible
Aside: Decorations

- Decorating a node
  - associating a value to it

- Two approaches
  - Add new attribute to each node
    - ex: `node.numDescendants = 5`
  - Maintain dictionary that maps nodes to decoration
    - do this if you can’t modify tree
    - ex: `descendantDict[node] = 5`
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Tree Traversals

- How would you enumerate every item in an array?
  - use a for loop from $i$ to $n$ and read $\text{Arr}[i]$

- How would you enumerate every item in a (linked) Tree?
  - not obvious…
  - because Trees don’t have an “obvious” order like arrays

- Tree traversal
  - algorithm that visits every node of a tree

- Many possible tree traversals
  - each kind of traversal visits nodes in different order
Breadth- vs. Depth-First Traversals
Traversals Strategy

- Why can we use a `for` loop to enumerate items in an array?
- Can we use a `for` loop to visit nodes in a *linked* Tree?
  - Why not?
  - We usually don't know how many nodes
  - Not clear what we should do at every iteration
- For tree traversals we’ll use a `while` loop
function \textit{traversal} (\textbf{root}): 
  Store root in S
  while S is not empty
    get node from S
    do something with node
    store children in S
Traversing Strategy

What is \( S \) exactly?
- A place we store nodes until we can process them

Which node of \( S \) should we process next?
- the first? the last?

```python
function traversal(root):
    Store root in S
    while S is not empty
        get node from S
        do something with node
        store children in S
```

Diagram:
```
  A
 / \   /
B   C F
   \  /
    D E
     \  
      H J
```
Traversing a tree structure:

**Traversing Strategy — Grab Oldest Node**

```plaintext
function traversal(root):
    Store root in S
    while S is not empty
        get node from S
        do something with node
    store children in S
```

```
A

B

C

D

E

F

G

H

I

A  B  C  D  E  F  G  H  I
```

```
S

A

B

C

D

E

F

G

H

I
```

Diagram of tree structure with nodes and edges.
function `traversal(root)`:  
    Store root in S  
    while S is not empty  
        get node from S  
        do something with node  
    store children in S

Does S remind you of something?
Traversals Strategy — Grab Oldest Node

- If we grab the oldest node in $S$
  - we're doing FIFO...
  - so $S$ is a queue!
- Traversal w/ Queue gives breadth-first traversal
- Why?
  - Queue guarantees a node is processed before its children
  - Children can be inserted in any order

```python
function bft(root):
    Q = new Queue()
    enqueue root
    while Q is not empty
        node = Q.dequeue()
        visit(node)
        enqueue node's left & right children
```
Breadth-First Traversal

- Start at root
  - Visit both of its children first,
    - Then all of its grandchildren,
      - Then great-grandchildren
    - etc…
- Also known as
  - level-order traversal
Depth-First Traversal

- What if we grab youngest node in \( S \)?
  - we’re doing LIFO…
  - so \( S \) is a stack!
  - Traversal w/ Stack gives us…
- Depth-first search
  - start from root
  - traverse each branch before backtracking
  - can produce different orders
Depth-First Traversal

function dft(root):
  S = new Stack()
  push root
  while S is not empty
    node = S.pop()
    visit(node)
    push node’s left & right children

- Why does Stack give DFT?
  - Stack guarantees entire branch will be visited before visiting another branch
  - Children can be pushed on stack in any order
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Recursive Depth-First Traversal

- DFT can be implemented recursively
- With recursion we can have 3 different orders
  - **pre-order**: visits node before visiting left and right children
  - **post-order**: visits each child before visiting node
  - **in-order**: visits left child, node and then right child
Depth-First Visualizations

Starting Pre Order Traversal...

Starting Inorder Traversal

Starting Post Order Traversal...
Pre-order Traversal

```
function preorder(node):
    visit(node)
    if node has left child
        preorder(node.left)
    if node has right child
        preorder(node.right)
```

Note: like iterative DFT
function **postorder**(node):
  if node has left child
    postorder(node.left)
  if node has right child
    postorder(node.right)
  visit(node)
In-order Traversal

function inorder(node):
    if node has left child
        inorder(node.left)
    visit(node)
    if node has right child
        inorder(node.right)
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When to Use What Traversal?

- How do you know which traversal to use?
- Sometimes it doesn’t matter
- Often one traversal makes solving problem easier
Tree Traversal Problem

Which traversal should be used to decorate nodes with # of descendants?

Activity #1

1 min
Which traversal should be used to decorate nodes with # of descendants?

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1 min
Which traversal should be used to decorate nodes with # of descendants?
Tree Traversal Problem

- Decorating with number of descendants?

- **Post-order**
  - visits both children before node
  - easy to calculate # of descendants if you know # of descendants of both children
  - try writing pseudo-code for this
Given root, which traversal should be used to test if tree is perfect?
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Tree Traversal Problem

- Testing if tree is perfect
- **Breadth-first**
  - traverses tree level by level
  - keep track of how many nodes at level
  - each level should have twice as many as previous level
Tree Traversal Problem

- Best traversal?
  - **post-order**: need to know size of subfolders before you can compute size of a folder
Tree Traversals Problems

- *Evaluate* arithmetic expression tree

\[(7 - (4 + 3)) + (9 / 3)\]  

- Best traversal?
  - **post-order:** to evaluate operation, you first need to evaluate sub-expression on each side
  - What should you do when you get to a leaf?
Tree Traversals Problems

- Best traversal?

  - **In-order:** gives nodes from left to right

Given tree, *print out* expression w/o parentheses

\[
\begin{align*}
\text{7} & \quad \text{4} & \quad \text{3} \\
& \quad + & \quad - \\
\text{4} & \quad \text{3} & \quad + \\
& \quad / & \quad / \\
\end{align*}
\]

\[
7 - 4 + 3 + 9 / 3
\]

- **Best traversal?**

  - **In-order:** gives nodes from left to right
Euler Tour Traversal

- Generic traversal of binary tree
  - pre-order, post-order and in-order are special cases
- Each node visited 3 times
  - left, bottom, right
Euler Tour Traversal

- Visit node on the
  - **left** → pre-order traversal
  - **bottom** → in-order traversal
  - **right** → post-order traversal
Euler Tour Traversal

function `eulerTour(node)`:  
- # pre-order
  `visitLeft(node)`

  if node has left child:
  `eulerTour(node.left)`

- # in-order
  `visitBelow(node)`

  if node has right child:
  `eulerTour(node.right)`

- # post-order
  `visitRight(node)`
Tree Traversal Problems

- Given tree, print out expression w/ parentheses
  \[ (7 - (4 + 3)) + (9 / 3) \]

- Best traversal?
  - **Euler tour**
Tree Traversal Problem

- Best traversal?
  - **Euler tour**

- Internal nodes
  - For pre-order/left visit, print "("
  - For in-order/bottom visit, print operator
  - For post-order/right visit, print ")"

- Leaves
  - Don’t do anything for pre-order/left and post-order/right visits
  - For in-order/bottom visit, print number
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Analyzing Binary Trees

- Many things can be modeled as binary trees
  - ex: Fibonacci recursive tree

\[ F(n) = F(n - 1) + F(n - 2) \]
Analyzing Binary Trees

- Knowing facts about binary trees can help with runtime analysis
  - ex: how many recursive calls are made by a binary recursive tree of height $n$?
- Perfect binary trees are easier to analyze...
  - ...so often we use them to estimate analysis of general trees
Analyzing Perfect Binary Trees

- Number of nodes in perfect binary tree of height $h$: $2^{h+1} - 1$
- Height of a perfect binary tree with $n$ nodes: $\log_2(n+1) - 1$
- Number of leaves in perfect binary tree of height $h$: $2^h$
- Number of nodes in perfect binary tree with $L$ leaves: $2L - 1$
Induction on Perfect Binary Trees

- Can use induction to prove things about PBTs
- Using recursive definition of perfect binary trees
- Tree $T$ is a perfect binary tree if
  - it has only one node
  - has root with left and right subtrees which are both perfect binary trees of same height
  - (if subtrees have height $h$, then $T$ has height $h+1$)
Example Inductive Proof on PBTs

- Prove $P(n)$:
  - number of nodes in a perfect binary tree of height $n$ is $f(n) = 2^{n+1} - 1$

- Base case $P(0)$:
  - number of nodes in perfect binary tree of height $0$ is $1$ (by definition)
  - $f(0) = 2^{0+1} - 1 = 2 - 1 = 1$

- Inductive hypothesis:
  - assume $P(k)$ is true (for some $k \geq 0$)
  - in words: the number of nodes in perfect binary tree of height $k$ is $f(k) = 2^{k+1} - 1$
Example Inductive Proof on PBTs

- Then prove that $P(k+1)$ is true:
  - Let $T$ be any perfect binary tree of height $k+1$
  - By definition, $T$ consists of root with two subtrees, $L$ and $R$, which are both perfect binary trees of height $k$
  - By inductive hypothesis, $L$ and $R$ both have $2^{k+1}-1$ nodes
  - So total number of nodes in $T$ is:
    - $2 \times (2^{k+1}-1) + 1 = 2^{k+2} - 2 + 1 = 2^{(k+1)+1} - 1$
  - Since we've proved
    - $P(0)$ is true
    - $P(k)$ implies $P(k+1)$ (for any $k \geq 0$)
    - It follows by induction that $P(n)$ is true for all $n \geq 0$
Tree ADT vs. Data Structure

- Is a Tree an ADT or a data structure?
  - It’s both
  - The answer depends on the context
- Trees are useful and interesting *abstract* objects
  - that capture parent/child relationships
  - they can be implemented using different data structures
    - some trees can be implemented using arrays
    - they can also be implemented using dictionaries
- But when computer scientists talk about Trees they often mean
  - the “linked tree” data structure
  - trees that are implemented using nodes and pointers