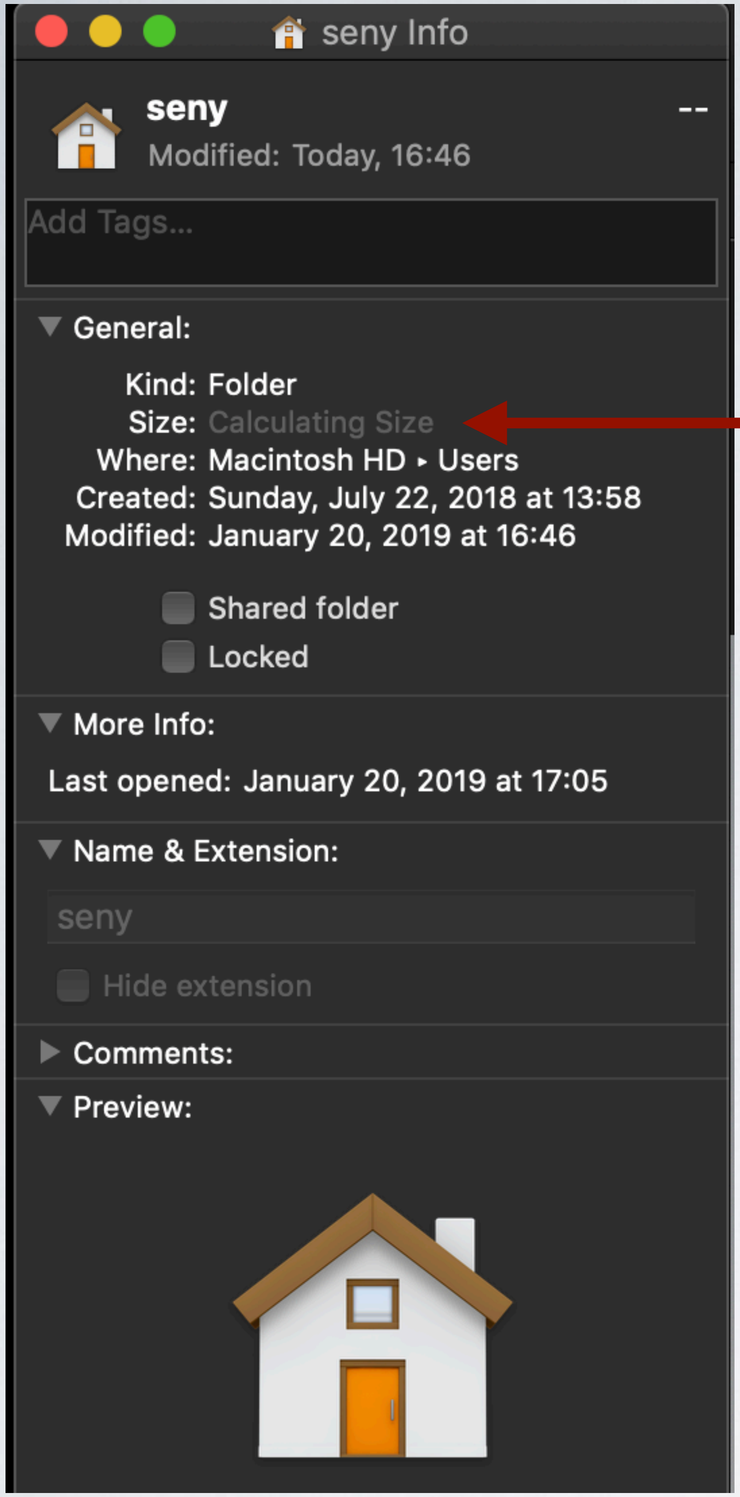
Tree Properties & Traversals

CS16: Introduction to Data Structures & Algorithms

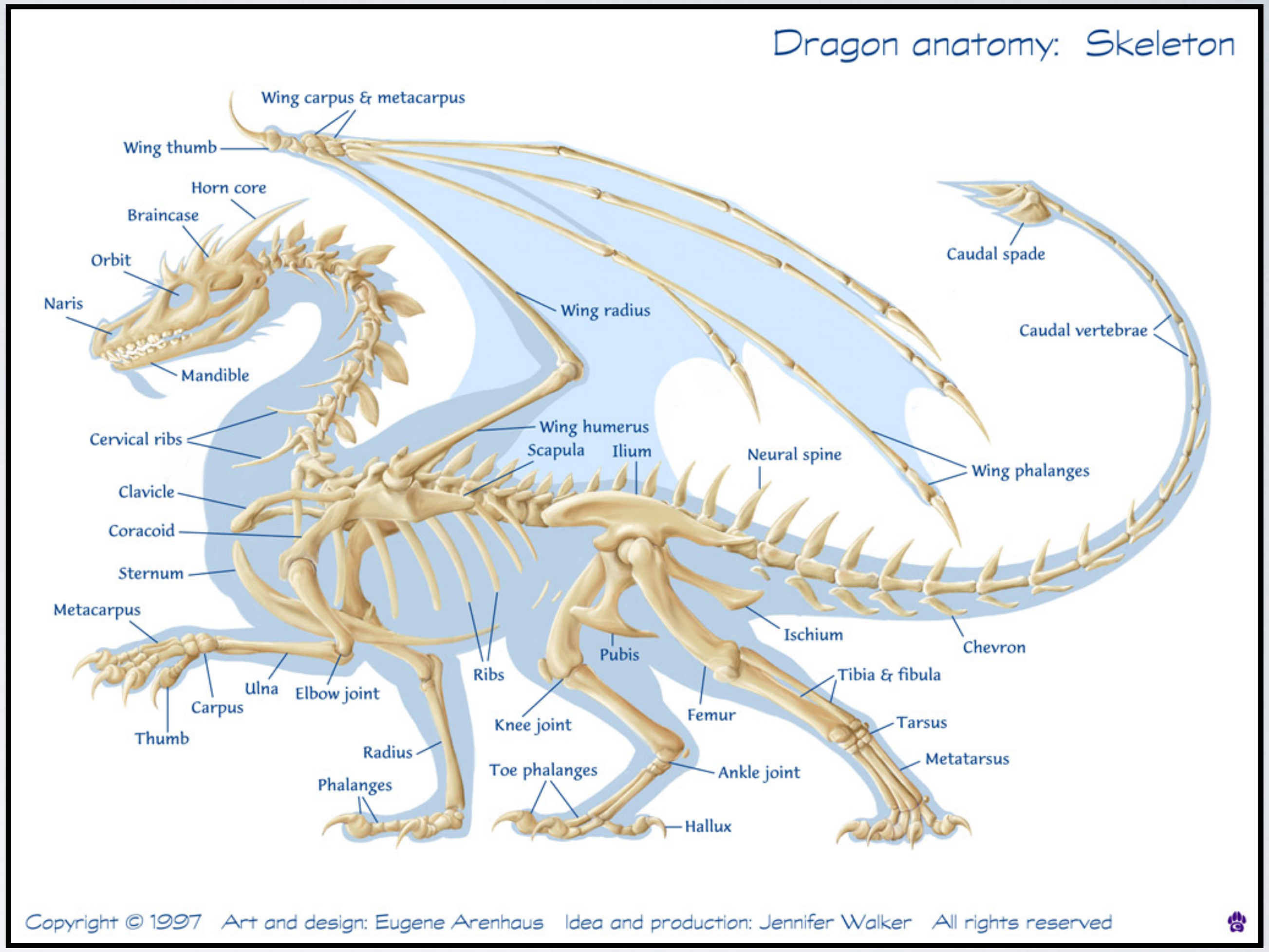
Spring 2020



- How does OS calculate size of directories?

Outline

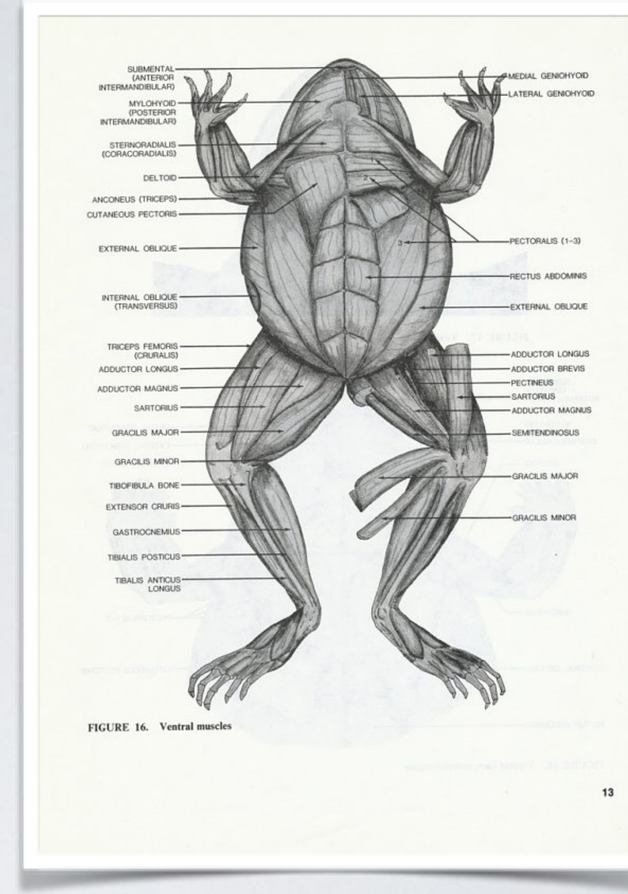
- Tree & Binary Tree ADT
- Tree Traversals
  - Breadth-First Traversal
  - Depth-First Traversal
- Recursive DFT
  - pre-order, post-order, in-order
- Euler Tour Traversal
- Traversal Problems
- Analysis on perfect binary trees

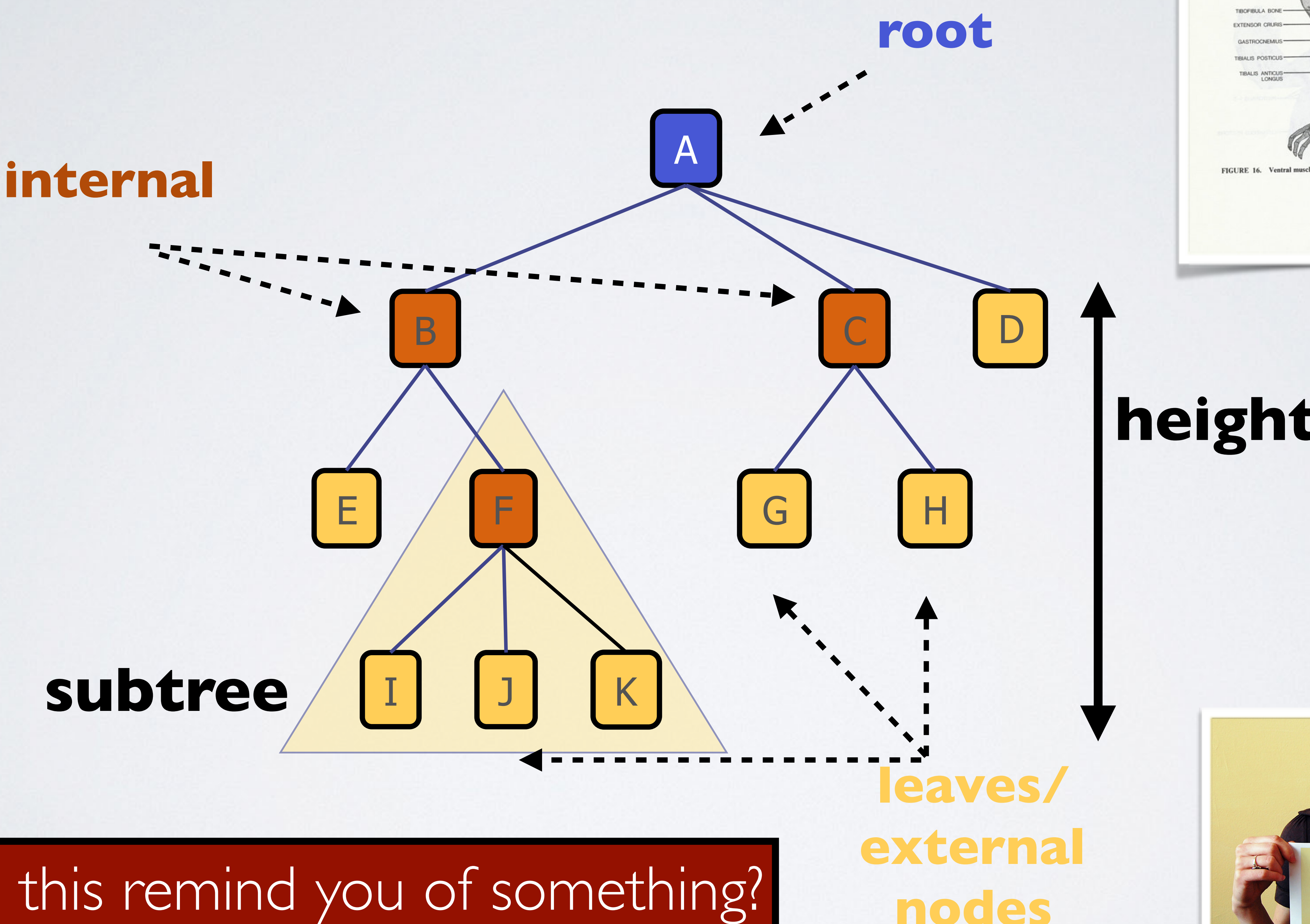


What is a Tree?

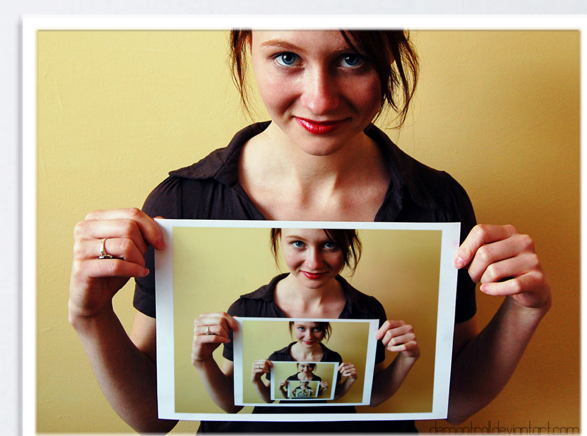
- Abstraction of hierarchy
- Tree consists of
  - nodes with parent/child relationship
- Examples
  - Files/folders (Windows, MacOSX, …, **CS33**)
  - Merkle Trees (Bitcoin, **CS166**)
  - Encrypted Data Structures (**CS2950-v**)
  - Datacenter Networks (Azure, AWS, Google, **CS168**)
  - Distributed Systems (Distributed Storage, Cluster computing, **CS138**)
  - AI & Machine Learning (Decision trees, **CS141**, **CS142**)

Tree "Anatomy"





Does this remind you of something?



Tree Terminology

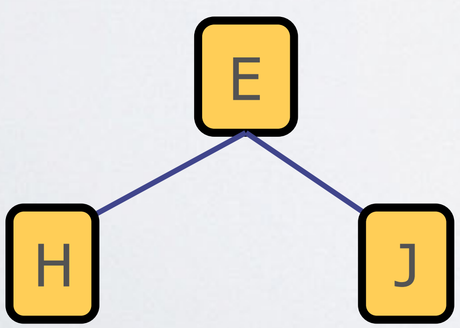
- **Root:** node without a parent (A)
- **Internal node:** node with at least one child (A, B, C, F)
- **Leaf (external node):** node without children (E, I, J, K, G, H, D)
- **Parent node:** node immediately above a given node (parent of C is A)
- **Child node:** node(s) immediately below a given node (children of C are G and H)
- **Ancestors of a node:**
  - parent, grandparent, grand-grandparent, etc. (ancestors of G are C, A)
- **Descendant of a node:** child, grandchild, grand-grandchild, etc.
- **Depth of a node:** number of ancestors (I has depth 3)
- **Height of a tree:**
  - maximum depth of any node (tree with just a root has height 0, this tree has height 3)
- **Subtree:** tree consisting of a node and its descendants

Tree ADT

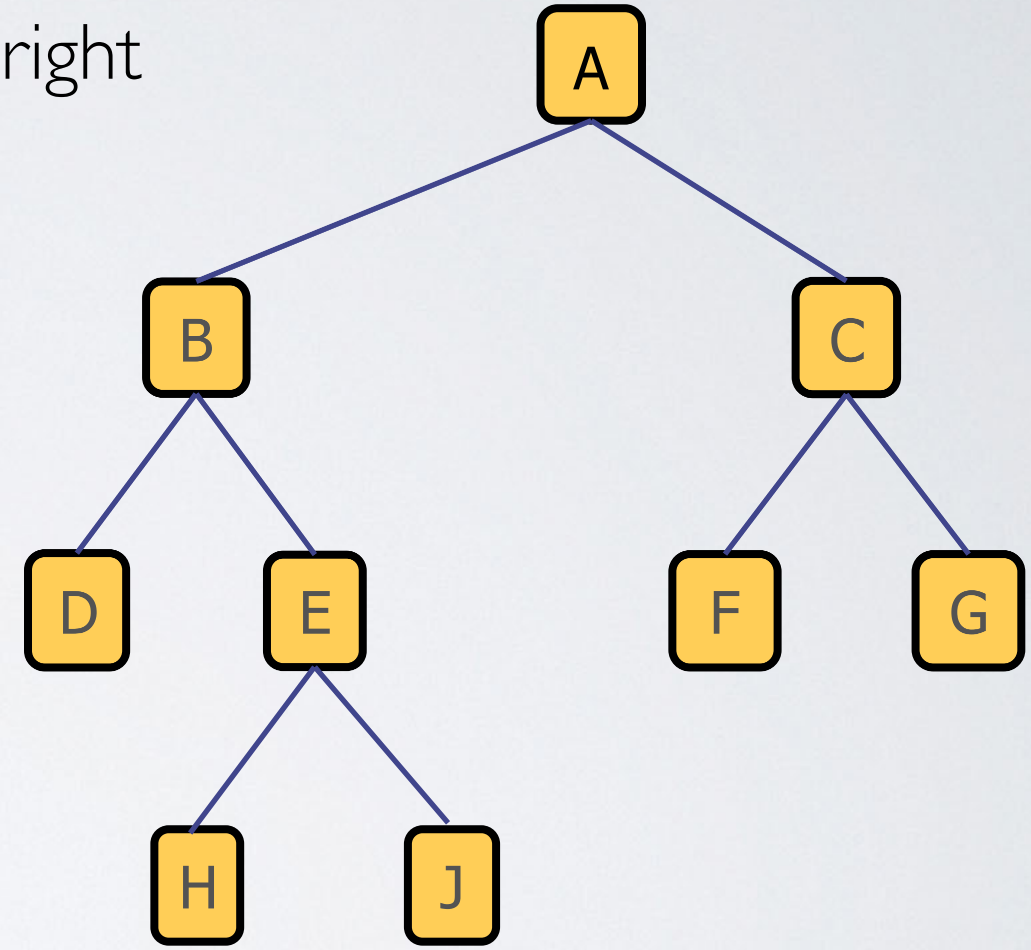
- Tree methods:
  - int **size**( ): returns the number of nodes
  - boolean **isEmpty**( ): returns true if the tree is empty
  - Node **root**( ): returns the root of the tree
- Node methods:
  - Node **parent**( ): returns the parent of the node
  - Node[ ] **children**( ): returns the children of the node
  - boolean **isInternal**( ): returns true if the node has children
  - boolean **isExternal**( ): returns true if the node is a leaf
  - boolean **isRoot**( ): returns true if the node is the root

Binary Trees

- Internal nodes have at most **2** children: left & right
  - if only **1** child, still need to specify if left or right
- *Recursive* definition of a Binary Tree
  - a single node
  - or a root node with at most 2 children
    - each of which is a binary tree
- Is 

a binary tree?
- Is 

a binary tree?

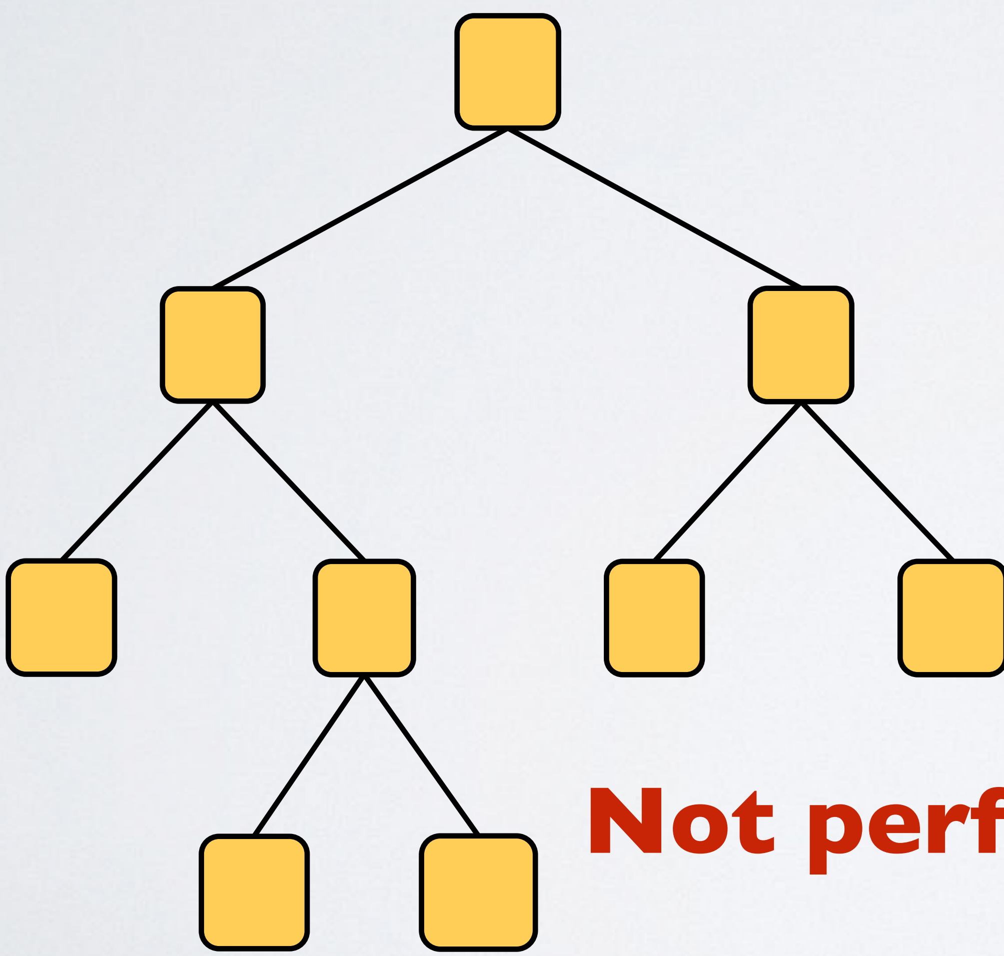


Binary Tree ADT

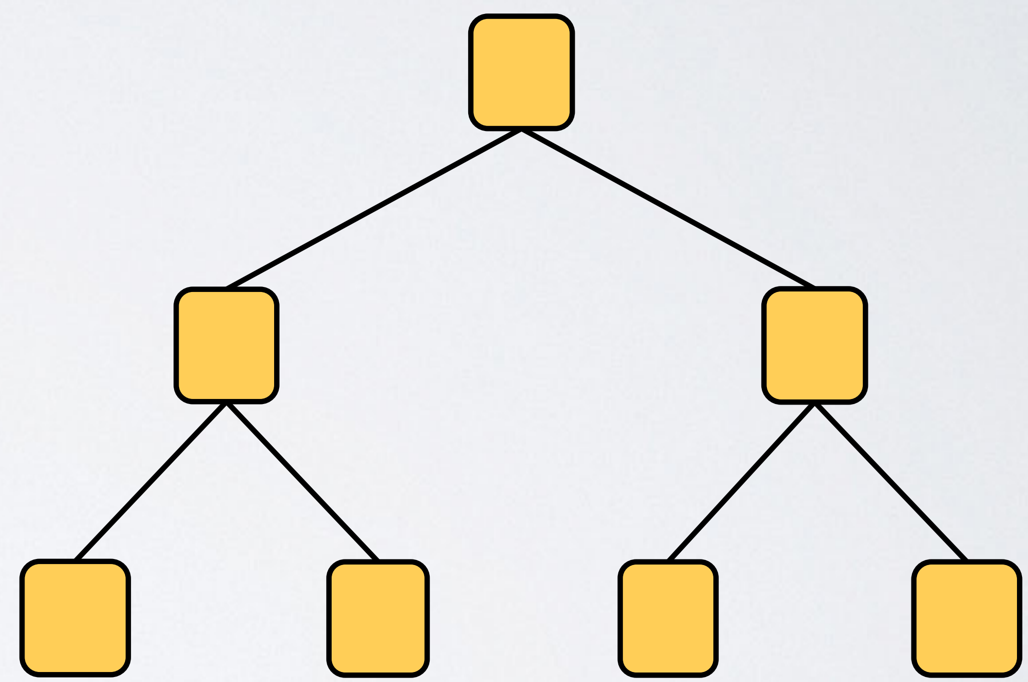
- In addition to Tree methods *binary trees* also support:
  - Node **`left`**( ): returns the left child if it exists, else NULL
  - Node **`right`**( ): returns the right child if it exists, else NULL
  - Node **`hasLeft`**( ): returns TRUE if node has left child
  - Node **`hasRight`**( ): returns TRUE if node has right child

Perfection

- A binary tree is *perfect* if
  - every level is completely full



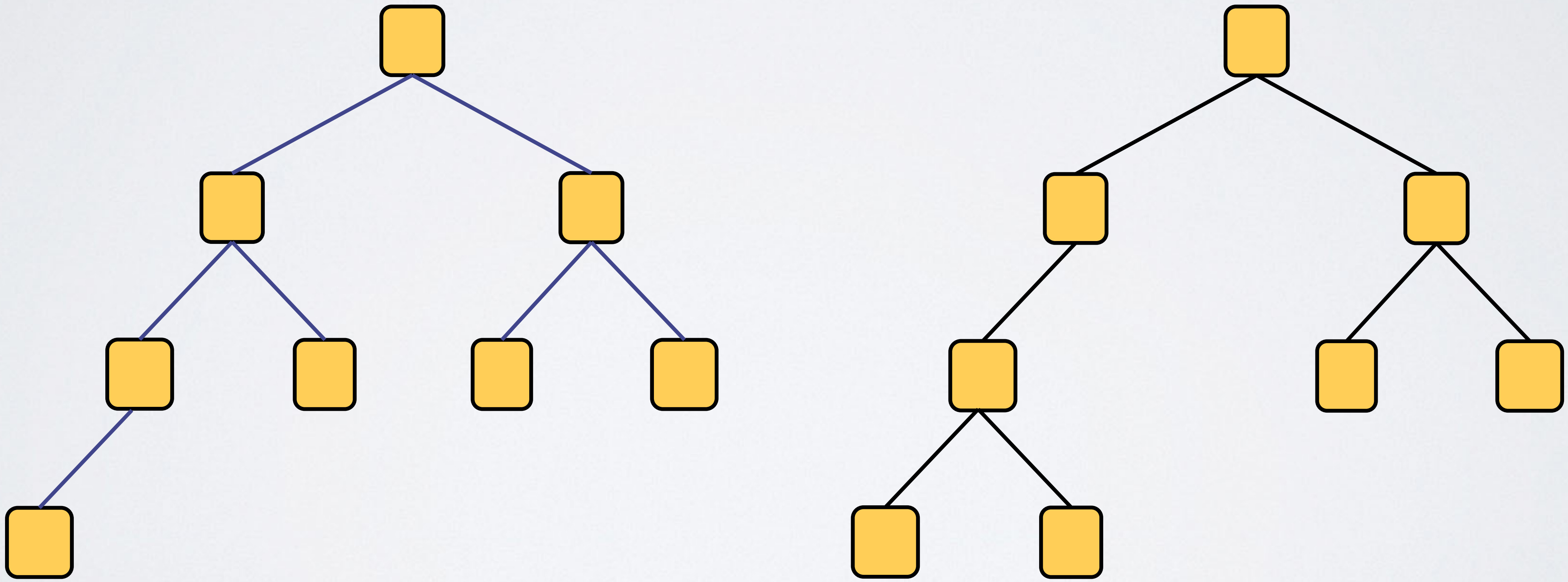
Not perfect



Perfect!

Completeness

- A binary tree is **left-complete** if
  - every level is completely full, possibly excluding the lowest level
  - all nodes are as far left as possible



Left-complete!

Not left-complete

Aside: Decorations

- Decorating a node
  - associating a value to it
- Two approaches
  - Add new attribute to each node
    - ex: `node.numDescendants = 5`
  - Maintain dictionary that maps nodes to decoration
    - do this if you can’t modify tree
    - ex: `descendantDict[node] = 5`

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- **Tree Traversals**
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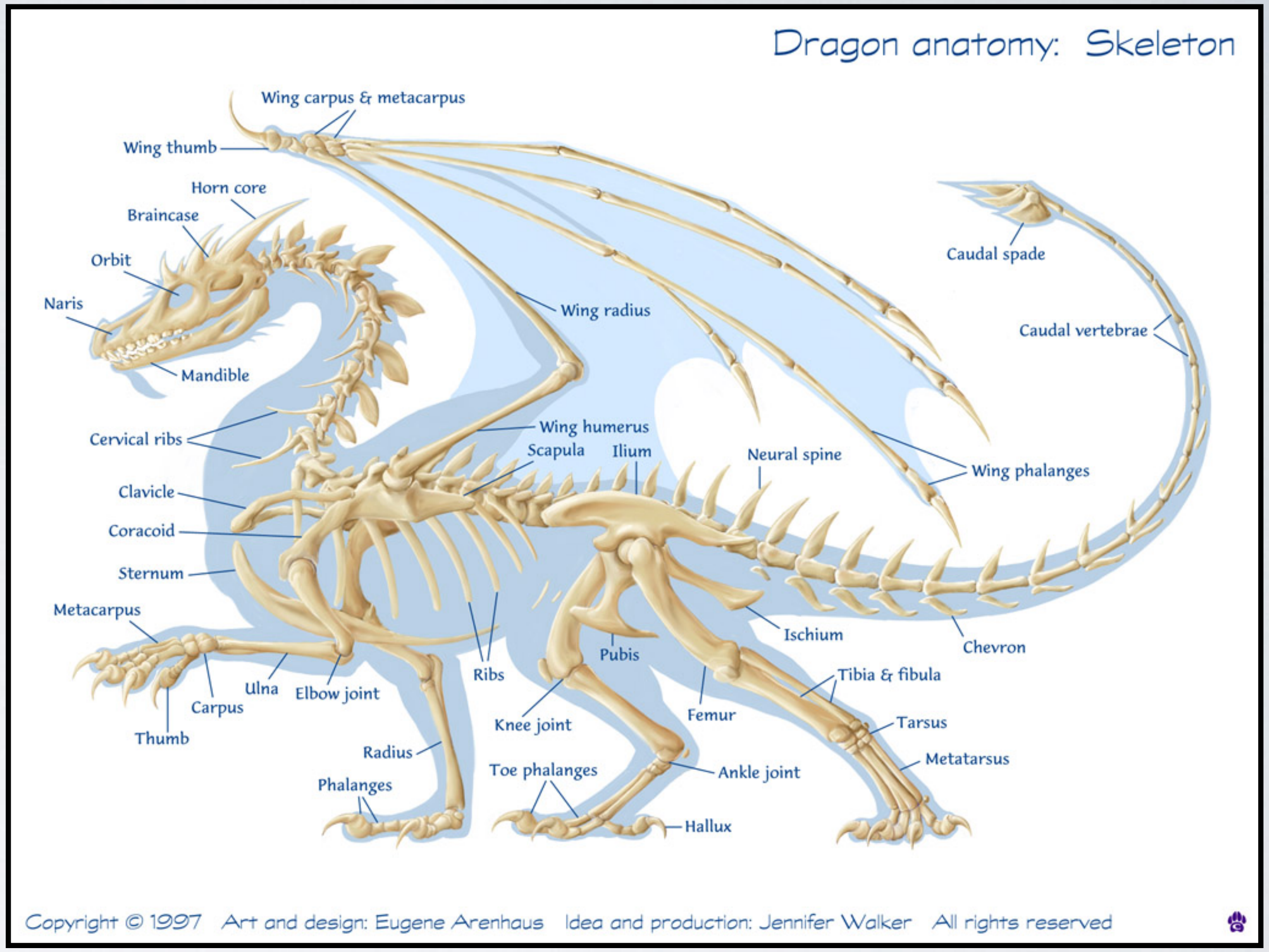
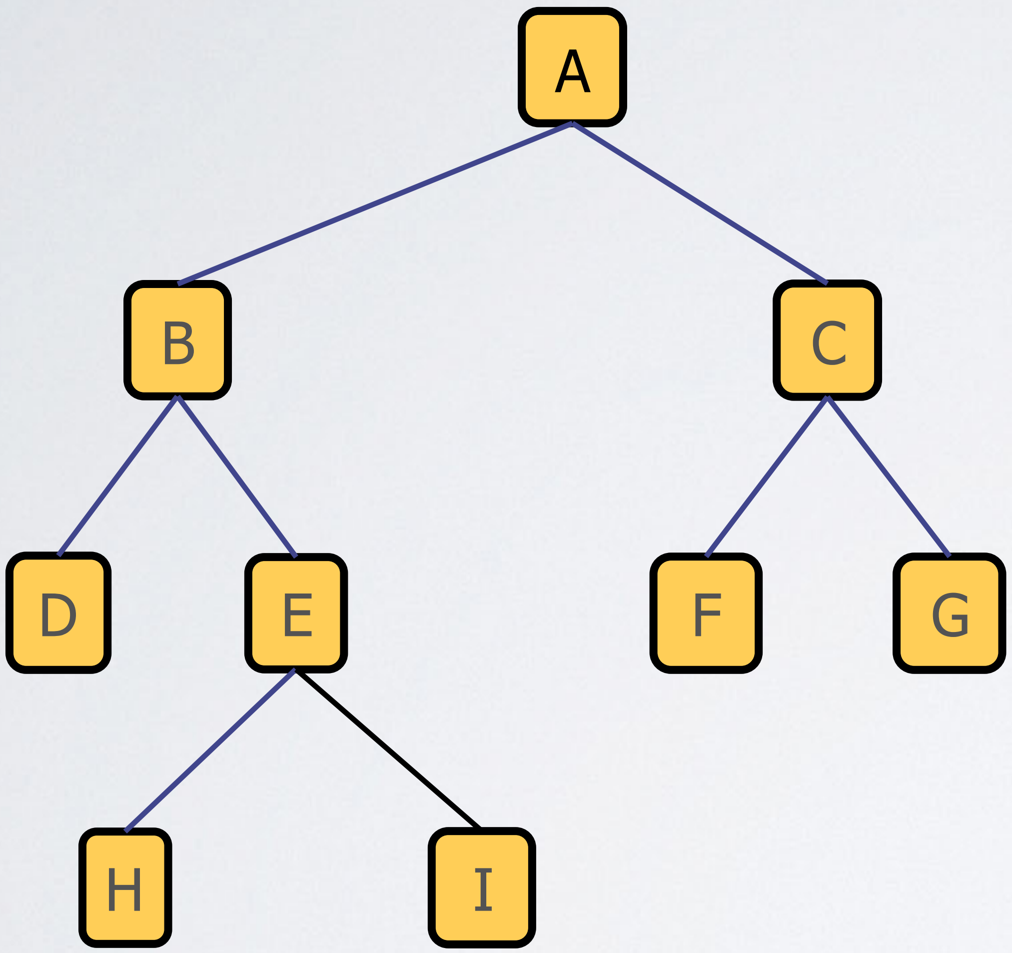
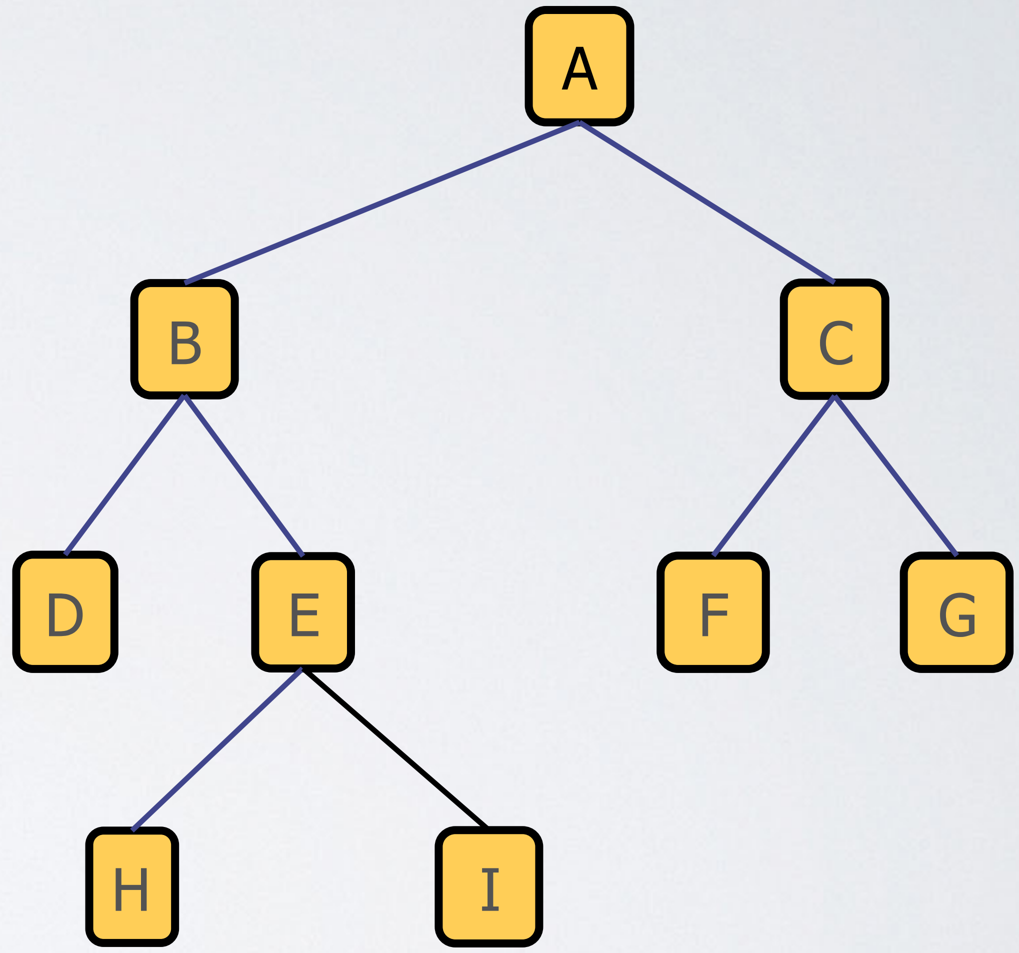


Tree Traversals

- How would you enumerate every item in an array?
  - use a for loop from **`i`** to **`n`** and read **`A[i]`**
- How would you enumerate every item in a (linked) Tree?
  - not obvious…
  - because Trees don't have an "obvious" order like arrays
- Tree traversal
  - algorithm that visits every node of a tree
- *Many* possible tree traversals
  - each kind of traversal visits nodes in different order

Breadth- vs. Depth-First Traversals



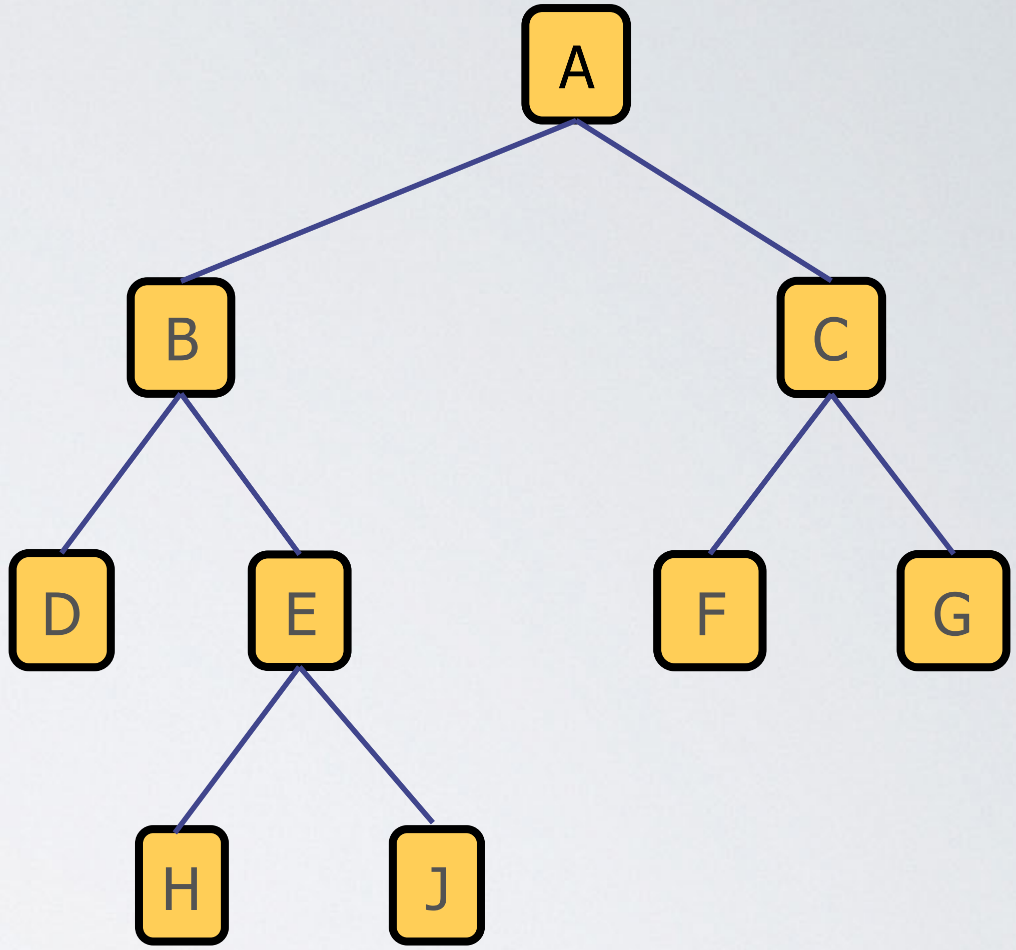


Traversal Strategy

- Why can we use a **`for`** loop to enumerate items in an array?
- Can we use a **`for`** loop to visit nodes in a *linked* Tree?
  - Why not?
  - we usually don't know how many nodes the tree has
  - not clear what we should do at every iteration
- For tree traversals we'll use a **`while`** loop

Traversal Strategy

```
function traversal(root):
  Store root in S
  while S is not empty
    get node from S
    do something with node
    store children in S
```



Traversal Strategy

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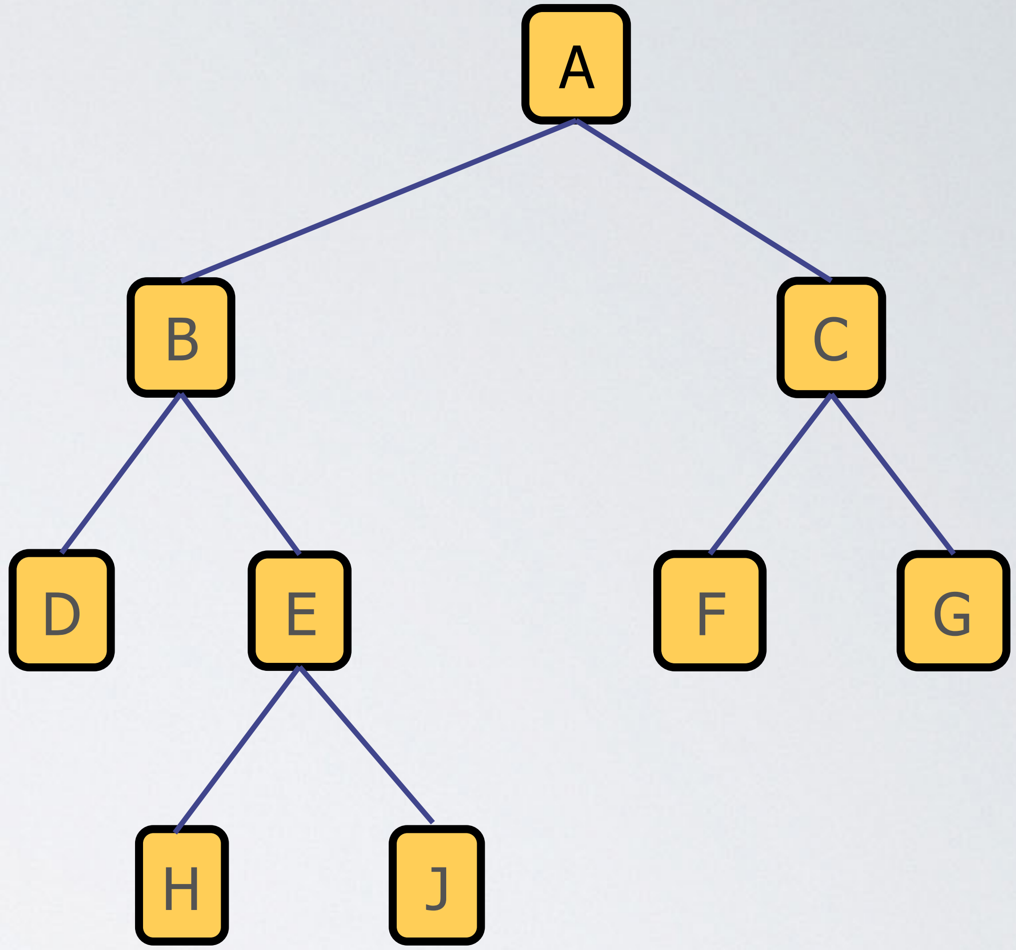

- What is **S** exactly?
  - A place we store nodes until we can process them
- Which node of **S** should we process next?
  - the first? the last?

Traversal Strategy — Grab Oldest Node

```
function traversal(root):
  Store root in S
  while S is not empty
    get node from S
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    store children in S
```

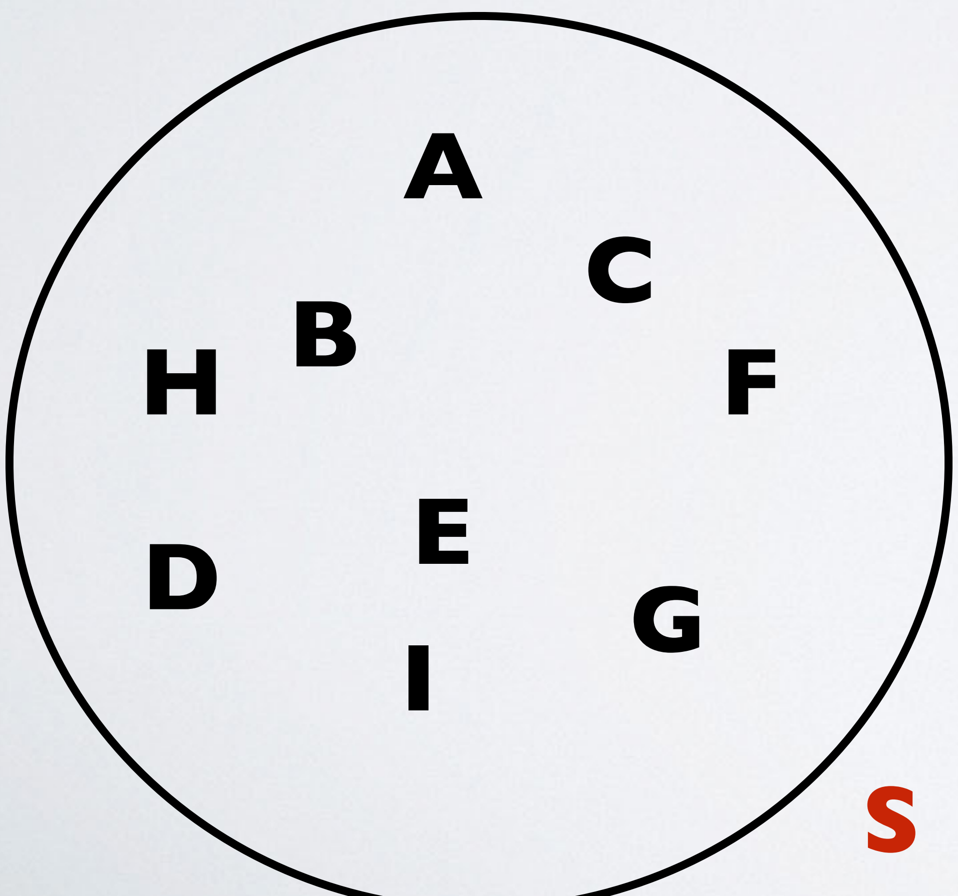

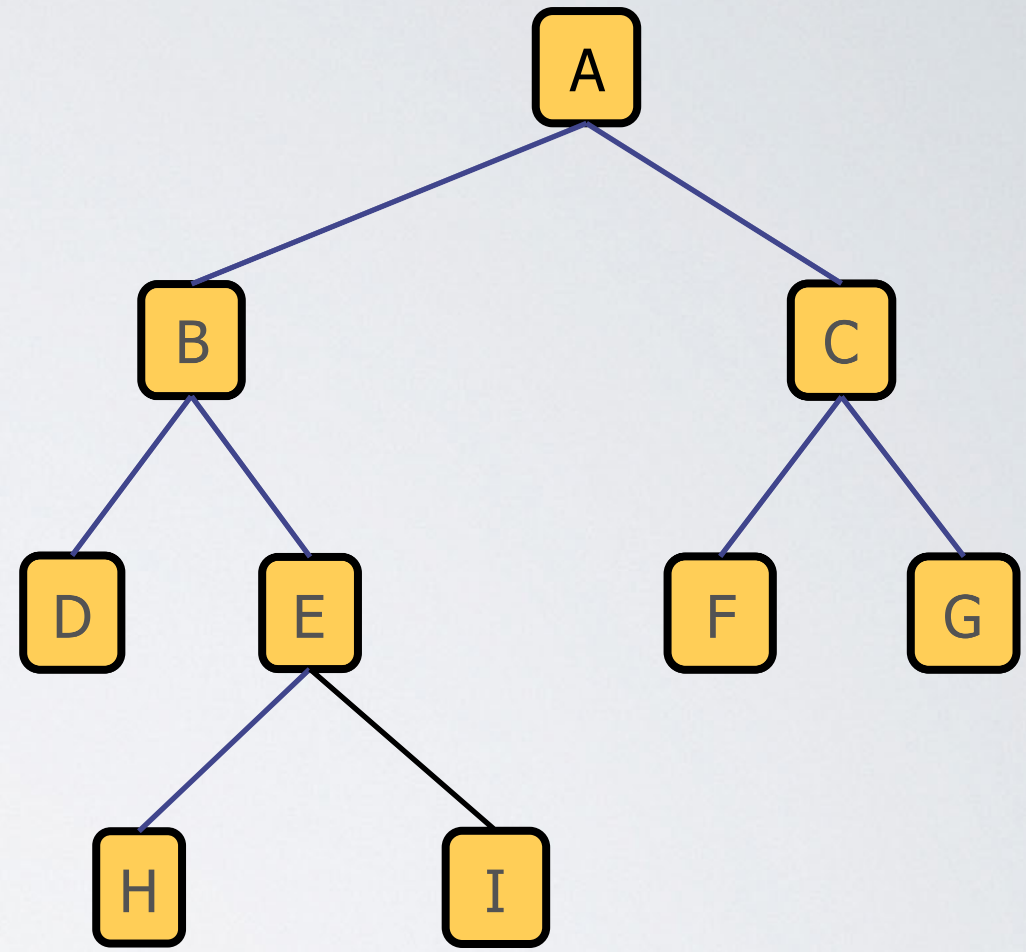

A B C D E F G H I

Traversal Strategy — Grab Oldest Node

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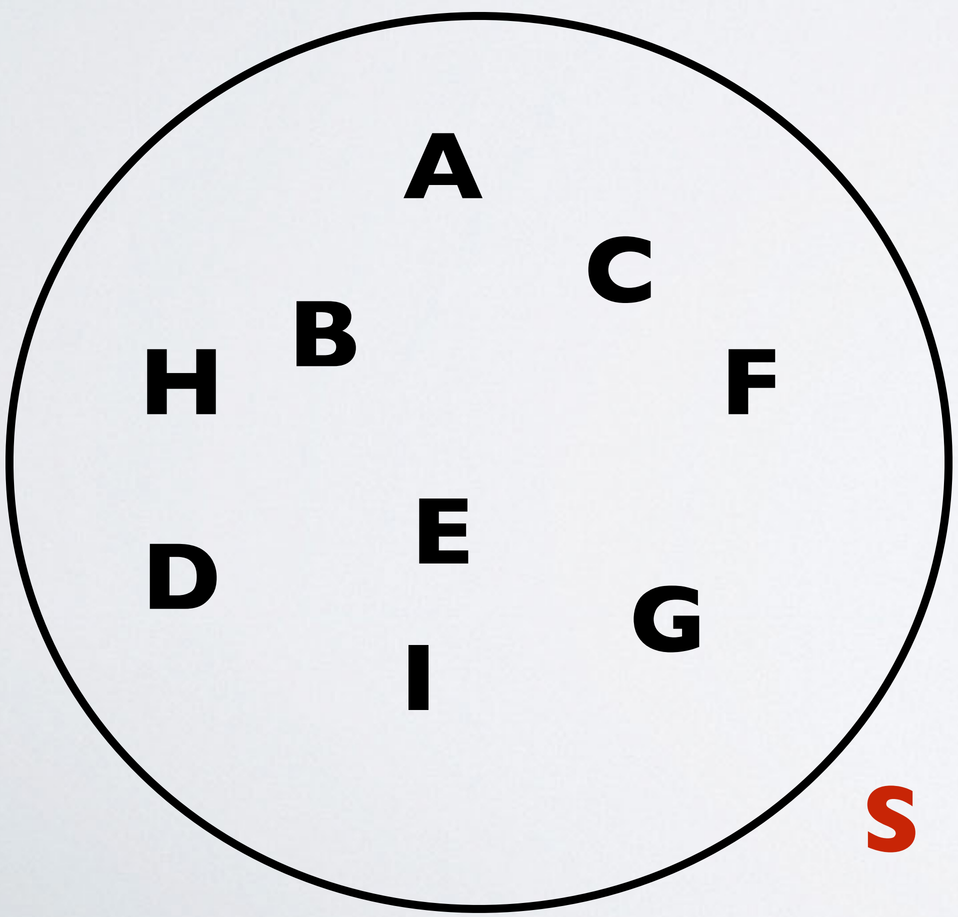
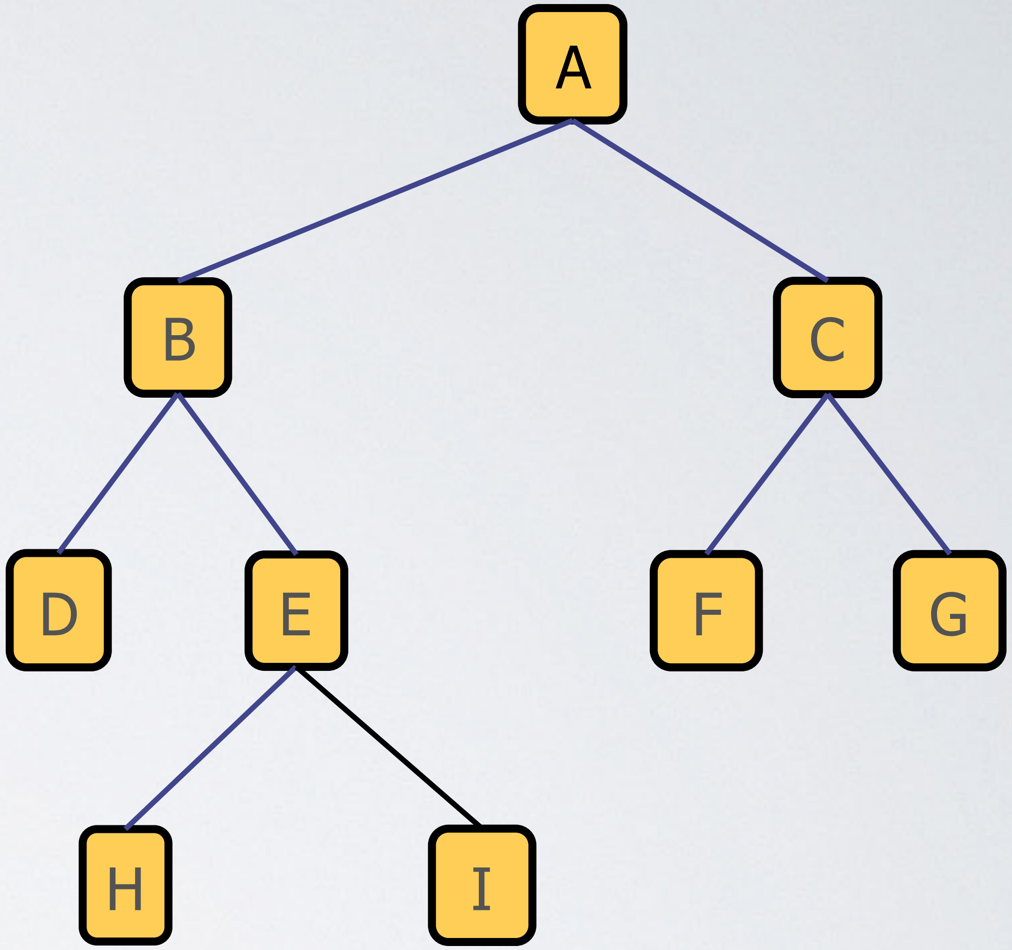




Does S remind you of something?

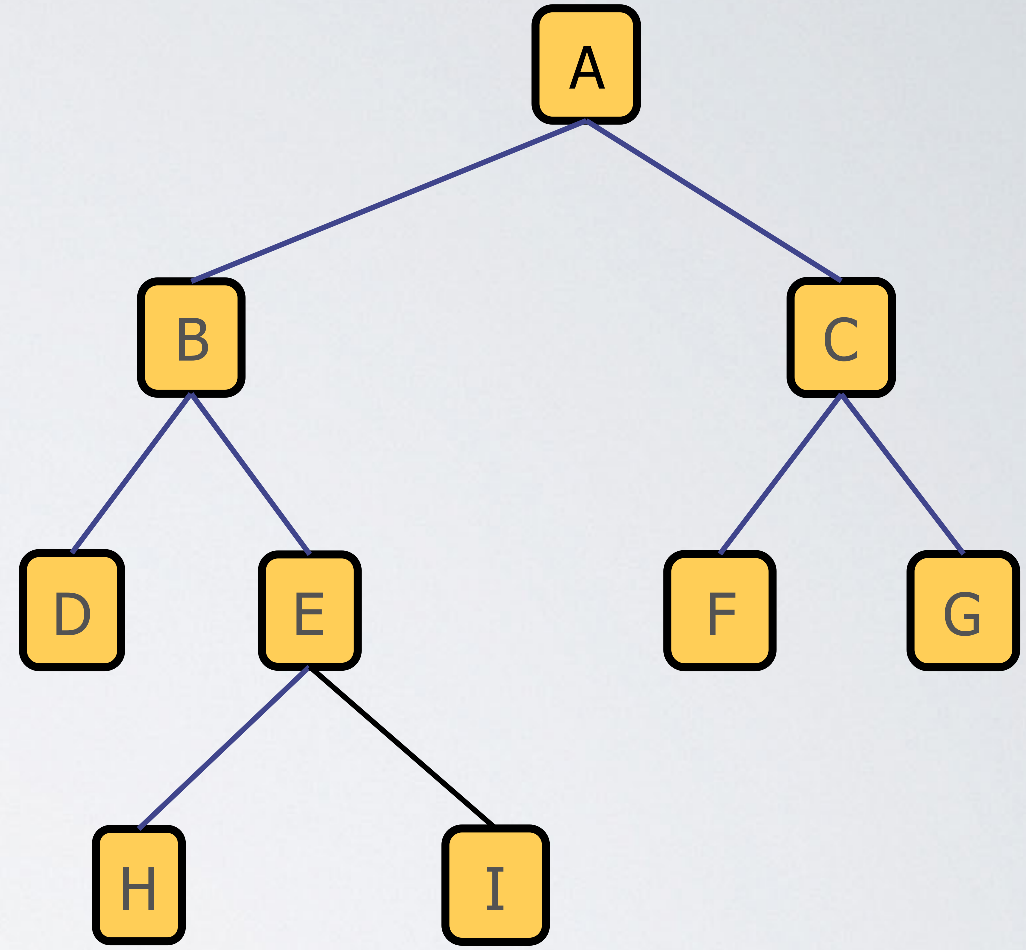
Traversal Strategy — Grab Oldest Node

```
function bft(root):
  Q = new Queue()
  enqueue root
  while Q is not empty
     node = Q.dequeue()
     visit(node)
     enqueue node's children
```

- If we grab the oldest node in **S**
  - we're doing FIFO…
  - so **S** is just a queue!
  - Traversal w/ Queue gives breadth-first traversal
- Why?
  - Queue guarantees a node is processed before its children
- Children can be inserted in any order

Breadth-First Traversal

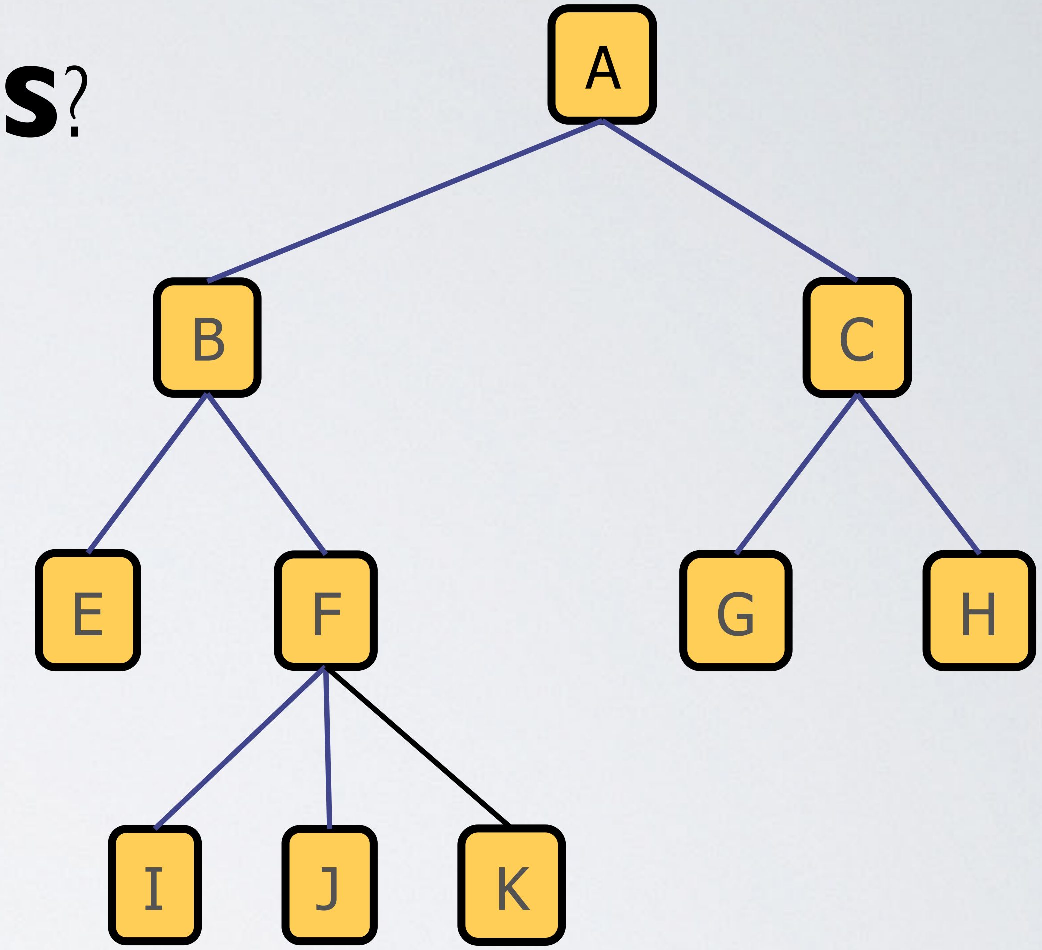
- Start at root
  - Visit both of its children first,
    - Then all of its grandchildren,
      - Then great-grandchildren
        - etc…
- Also known as
  - level-order traversal



A B C D E F G H I

Depth-First Traversal

- What if we grab youngest node in **S**?
  - we're doing LIFO…
  - so **S** is a stack!
  - Traversal w/ Stack gives us…
- Depth-first search
  - start from root
  - traverse each branch before backtracking
  - can produce different orders



A C H G B F K J I E

A B E F I J K C G H

Depth-First Traversal

```
function dft(root):
  S = new Stack()
  push root
  while S is not empty
    node = S.pop()
    visit(node)
    push node's children
```

- Why does Stack give DFT?
  - Stack guarantees a node's descendants will be visited before its sibling's descendants
- Children can be pushed on stack in any order

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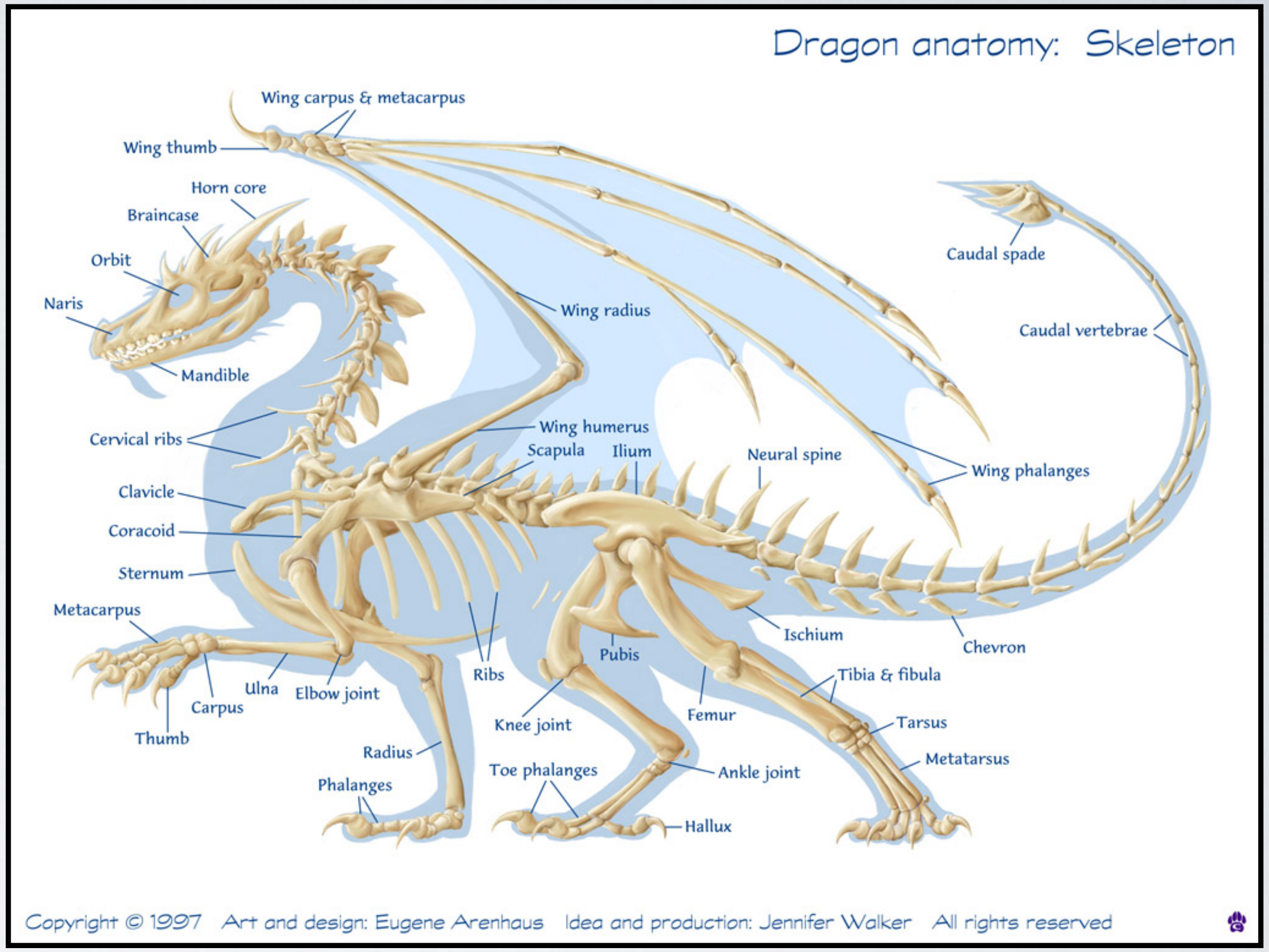
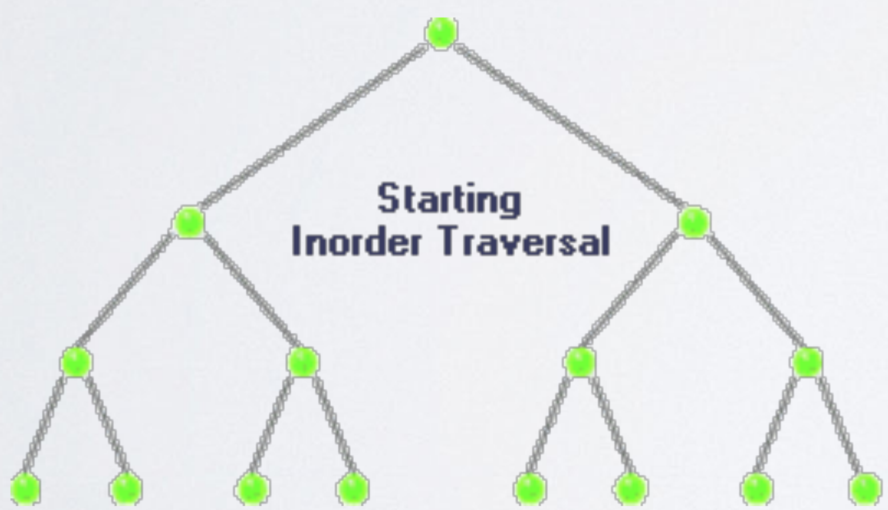


Recursive Depth-First Traversal

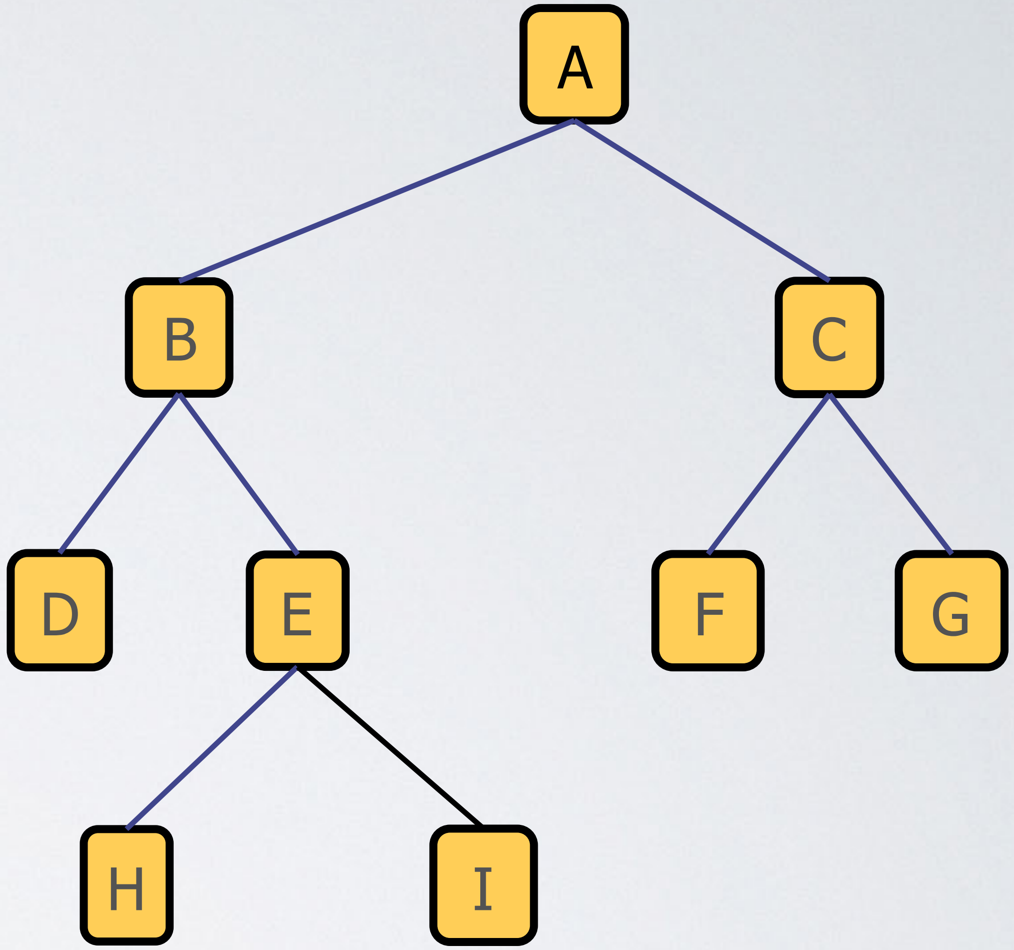
- DFT can be implemented recursively
- With recursion we can have *3* different orders
  - **pre-order:** visits node before visiting left and right children
  - **post-order:** visits each child before visiting node
  - **in-order:** visits left child, node and then right child

Depth-First Visualizations



Pre-order Traversal

```
function preorder(node):
  visit(node)
  if node has left child
    preorder(node.left)
  if node has right child
    preorder(node.right)
```

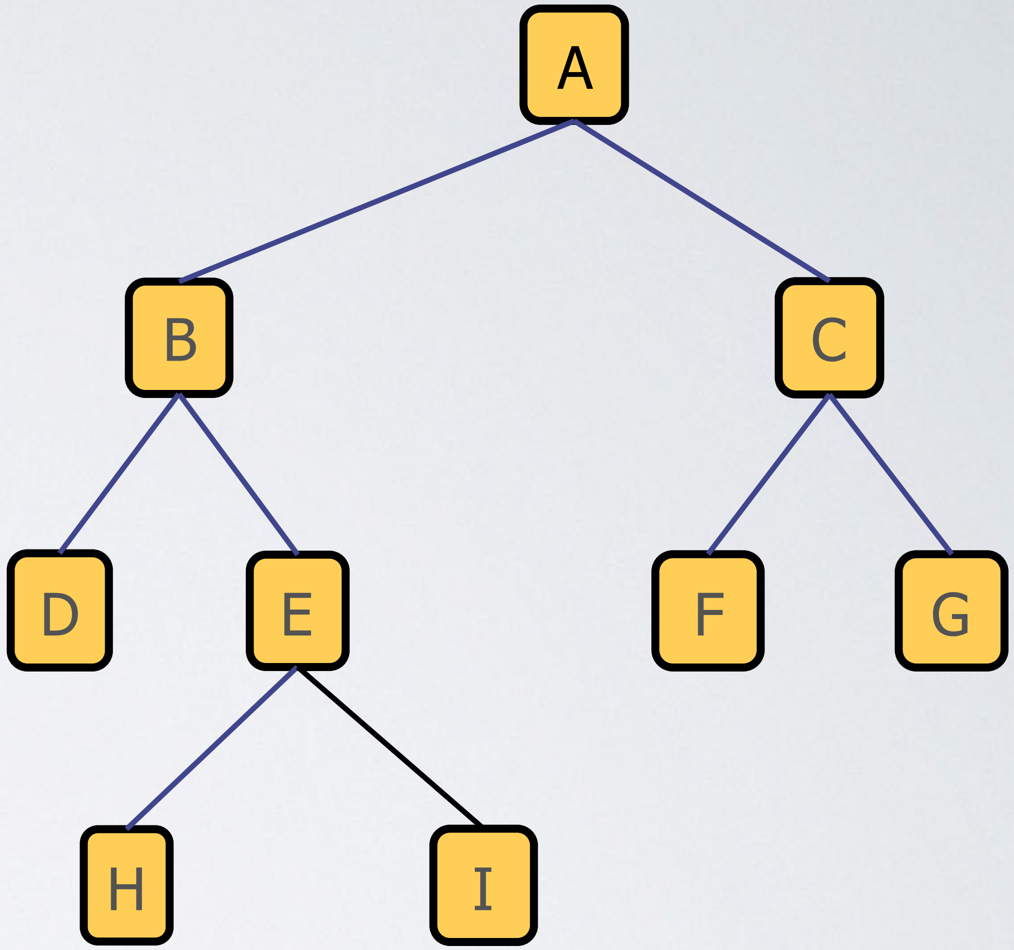


A B D E H I C F G

Note: like iterative DFT

Post-order Traversal

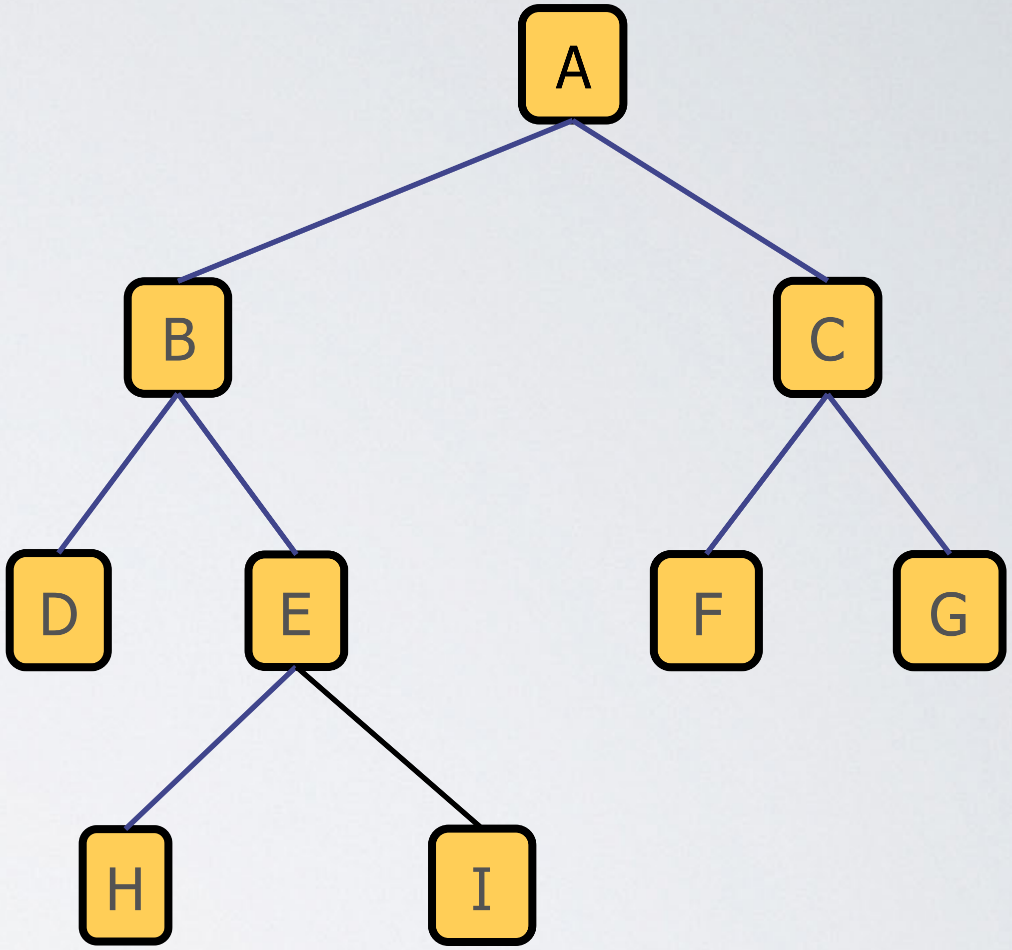
```
function postorder(node):
  if node has left child
    postorder(node.left)
  if node has right child
    postorder(node.right)
  visit(node)
```



D H I E B F G C A

In-order Traversal

```
function inorder(node):
  if node has left child
    inorder(node.left)
  visit(node)
  if node has right child
    inorder(node.right)
```



D B H E I A F C G

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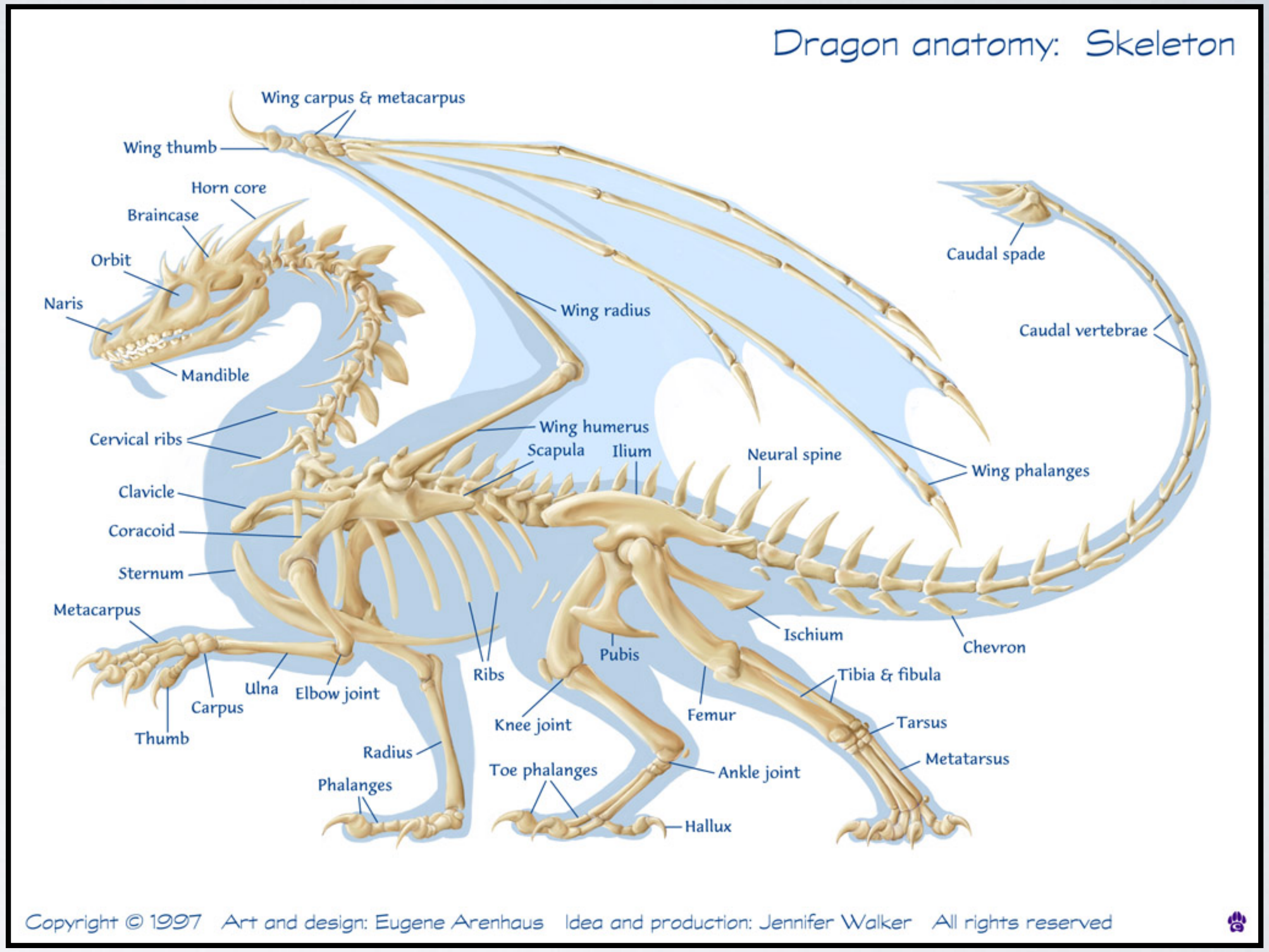


When to Use What Traversal?

- How do you know which traversal to use?
- Sometimes it doesn’t matter
- Often one traversal makes solving problem easier

Tree Traversal Problem

Which traversal should be used to decorate nodes with # of descendants?

Activity #1

1 min

Tree Traversal Problem

Which traversal should be used to decorate nodes with # of descendants?

Activity #1

1 min

Tree Traversal Problem

Which traversal should be used to decorate nodes with # of descendants?

Activity #1

0 min

Tree Traversal Problem

- Decorating with number of descendants?
- **Post-order**
  - visits both children before node
  - easy to calculate # of descendants if you know # of descendants of both children
  - try writing pseudo-code for this

Tree Traversal Problem

Given root, which traversal should be used to test if tree is perfect?

Activity #2

1 min

Tree Traversal Problem

Given root, which traversal should be used to test if tree is perfect?

Activity #2

1 min

Tree Traversal Problem

Given root, which traversal should be used to test if tree is perfect?

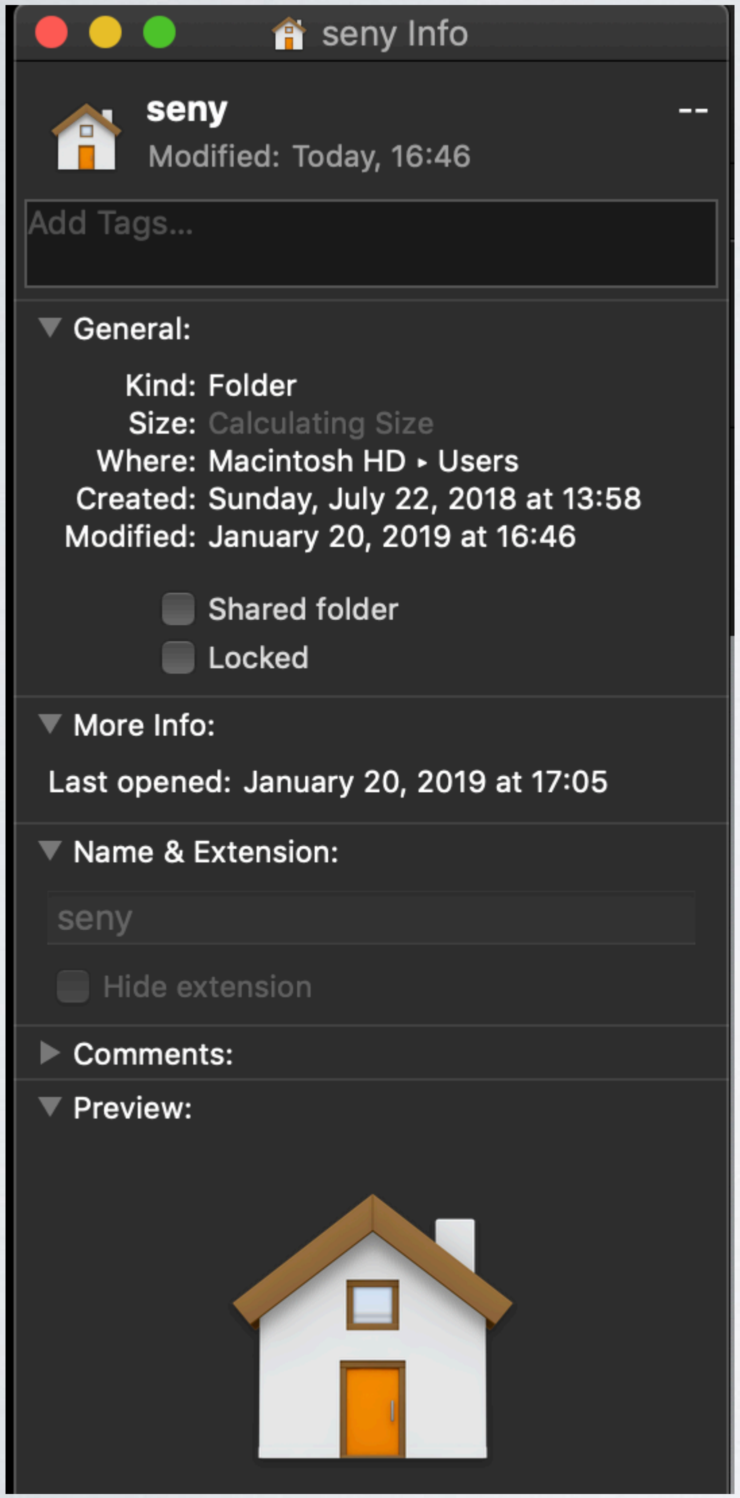
Activity #2

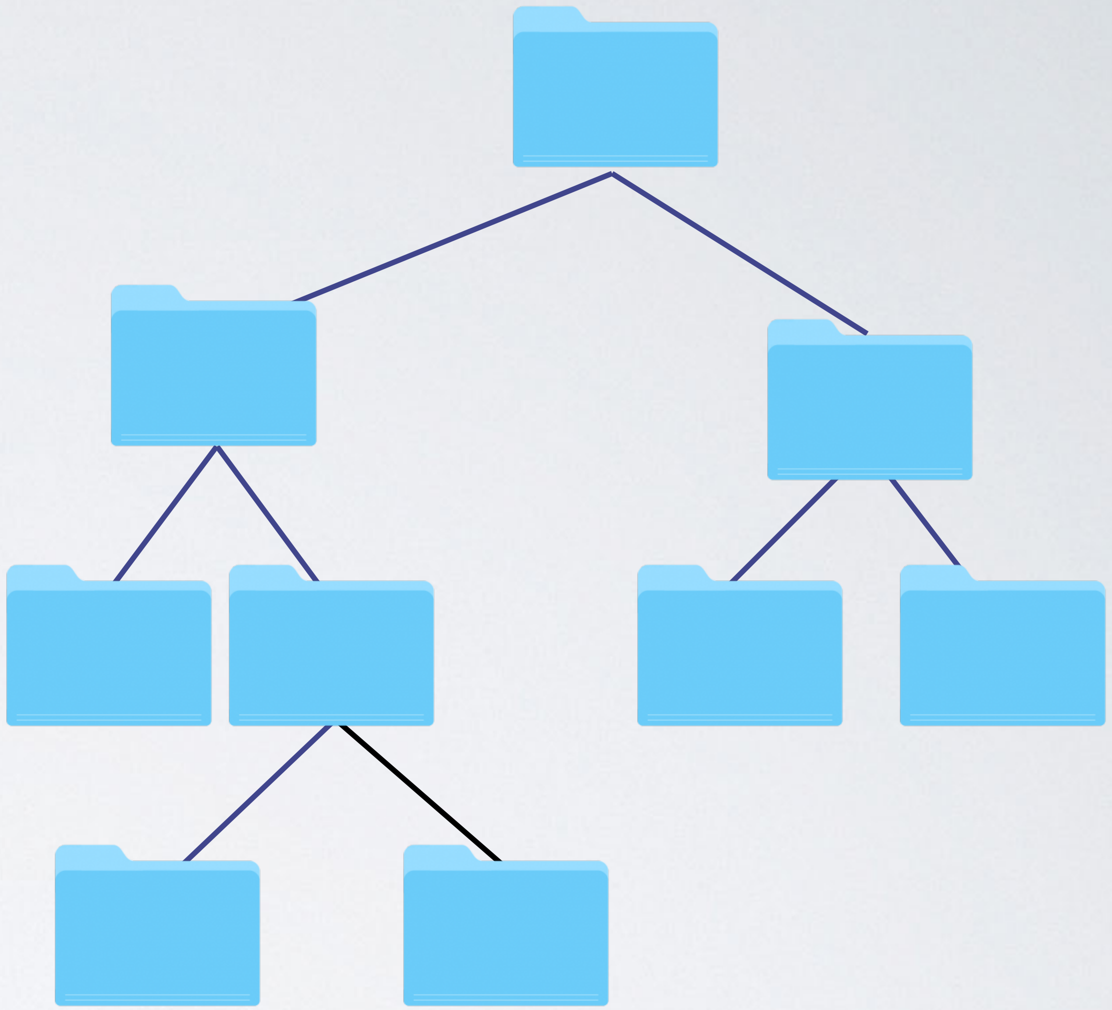
0 min

Tree Traversal Problem

- Testing if tree is perfect
- **Breadth-first**
  - traverses tree level by level
  - keep track of how many nodes at level
  - each level should have twice as many as previous level

Tree Traversal Problem



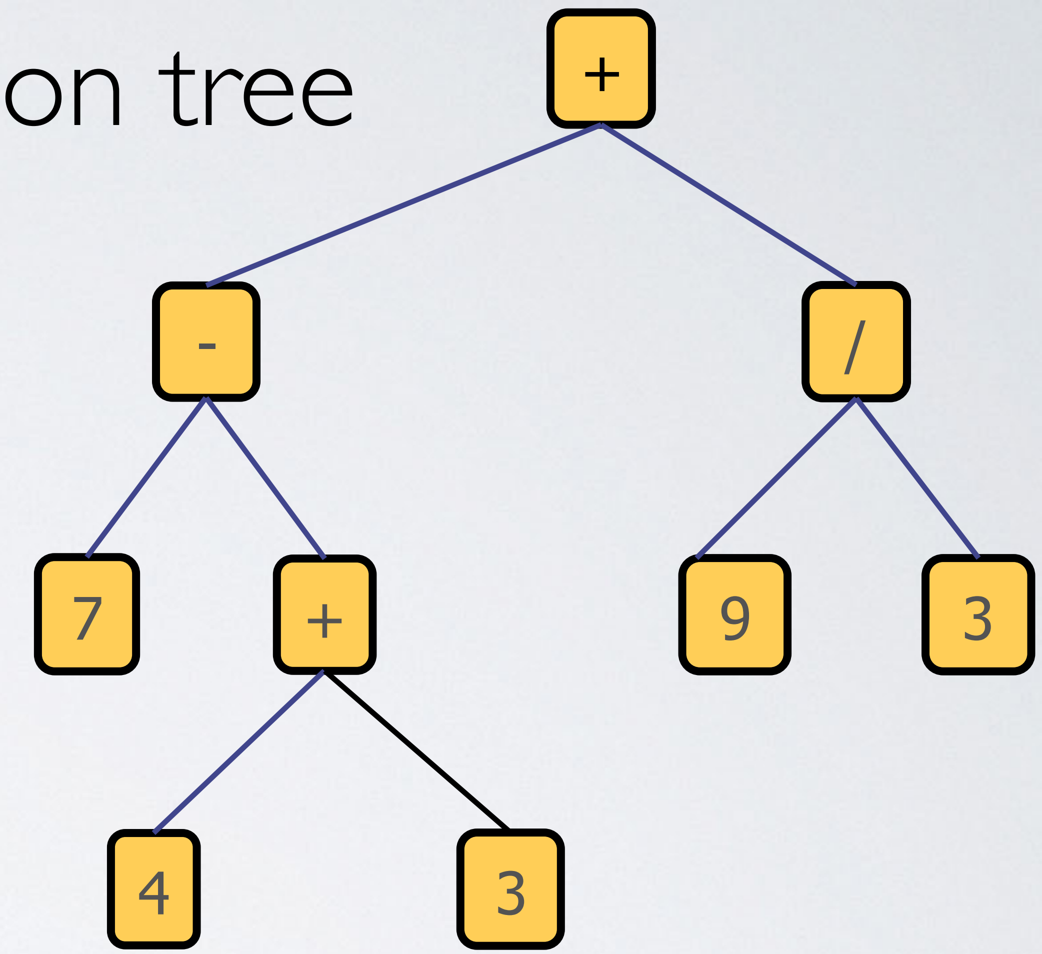


- Best traversal?
  - **post-order:** need to know size of subfolders before you can compute size of a folder

Tree Traversals Problems

- *Evaluate* arithmetic expression tree

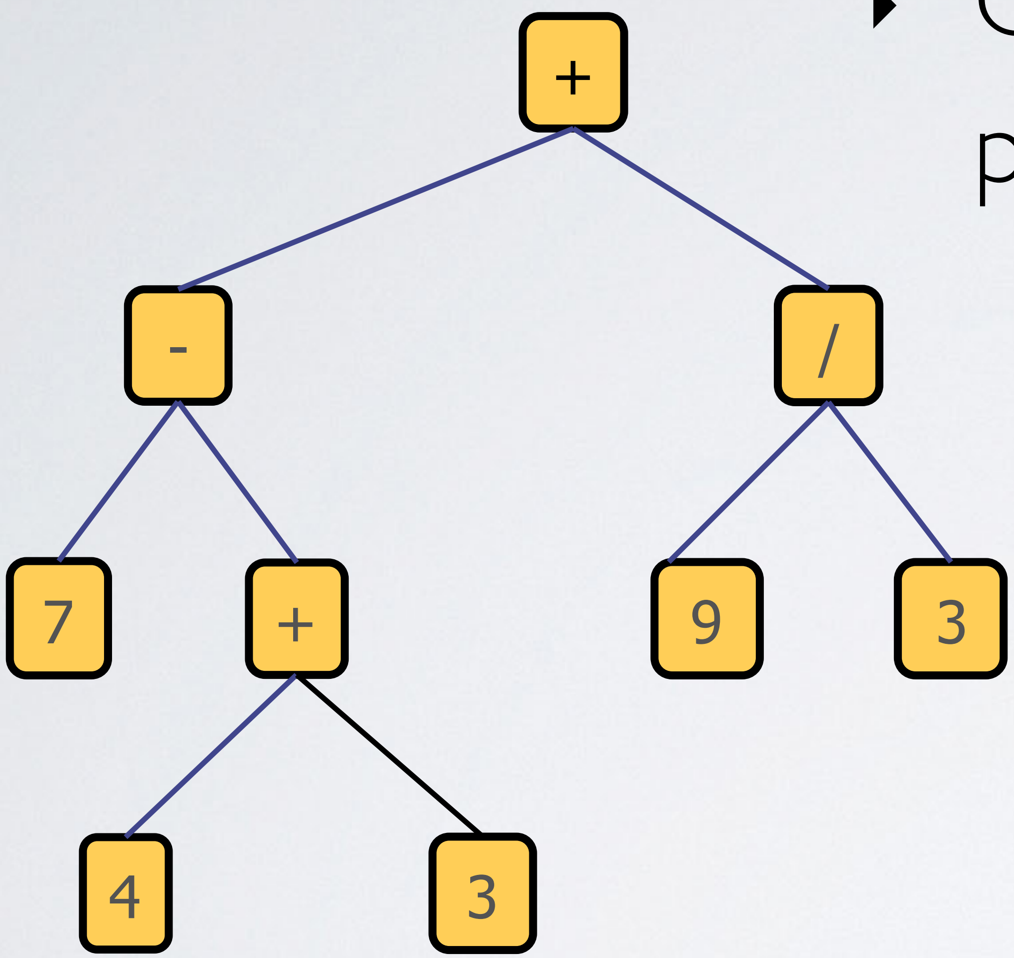
$(7 - (4 + 3)) + (9 / 3) \quad =$



- Best traversal?
  - **post-order:** to evaluate operation, you first need to evaluate sub-expression on each side
  - What should you do when you get to a leaf?

Tree Traversals Problems

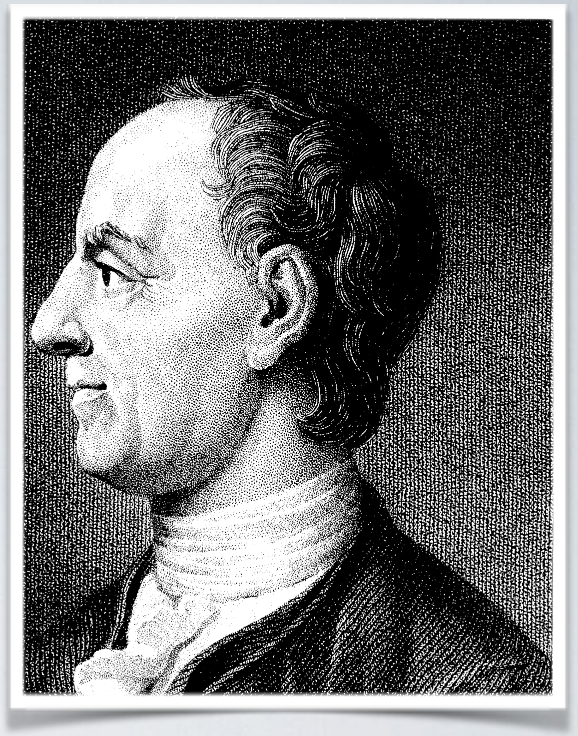
- Given tree, *print out* expression w/o parentheses



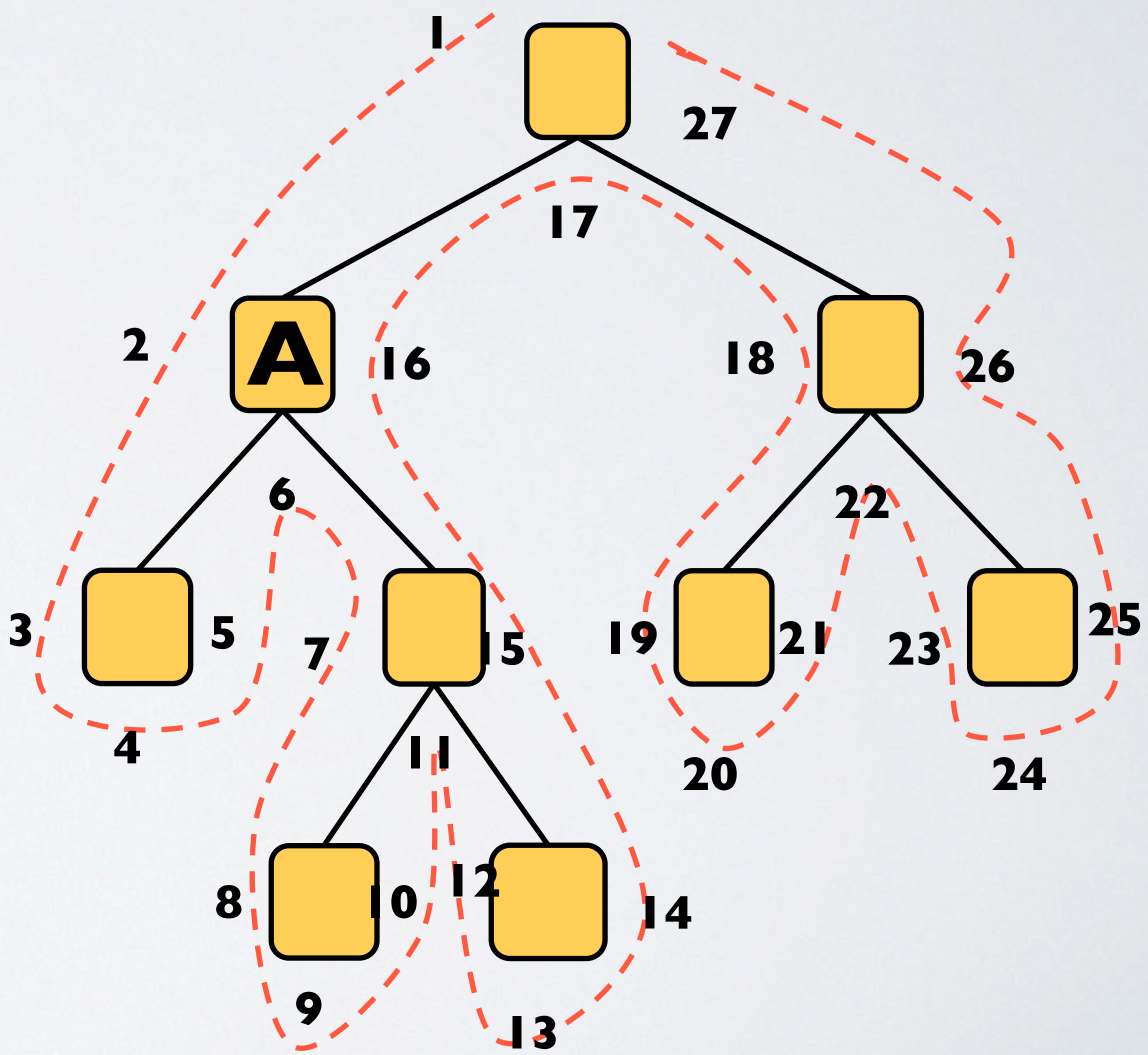
$= \quad 7 - 4 + 3 + 9 / 3$

- Best traversal?
  - **in-order:** gives nodes from left to right

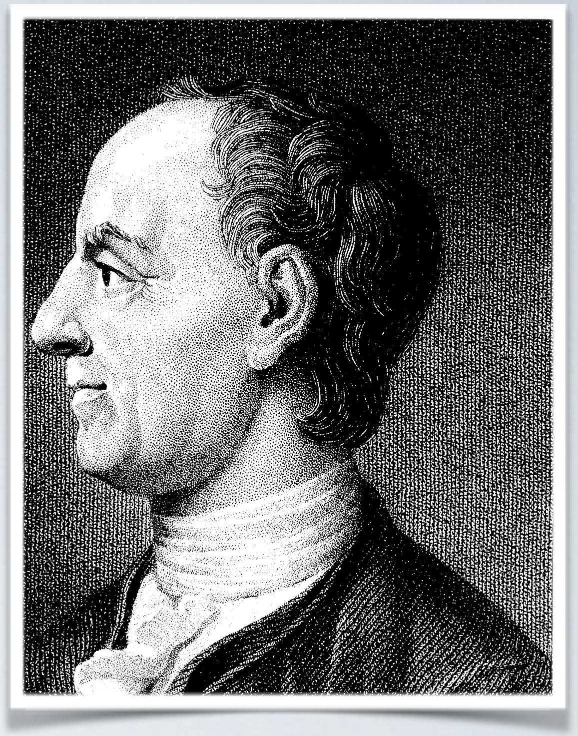
Euler Tour Traversal



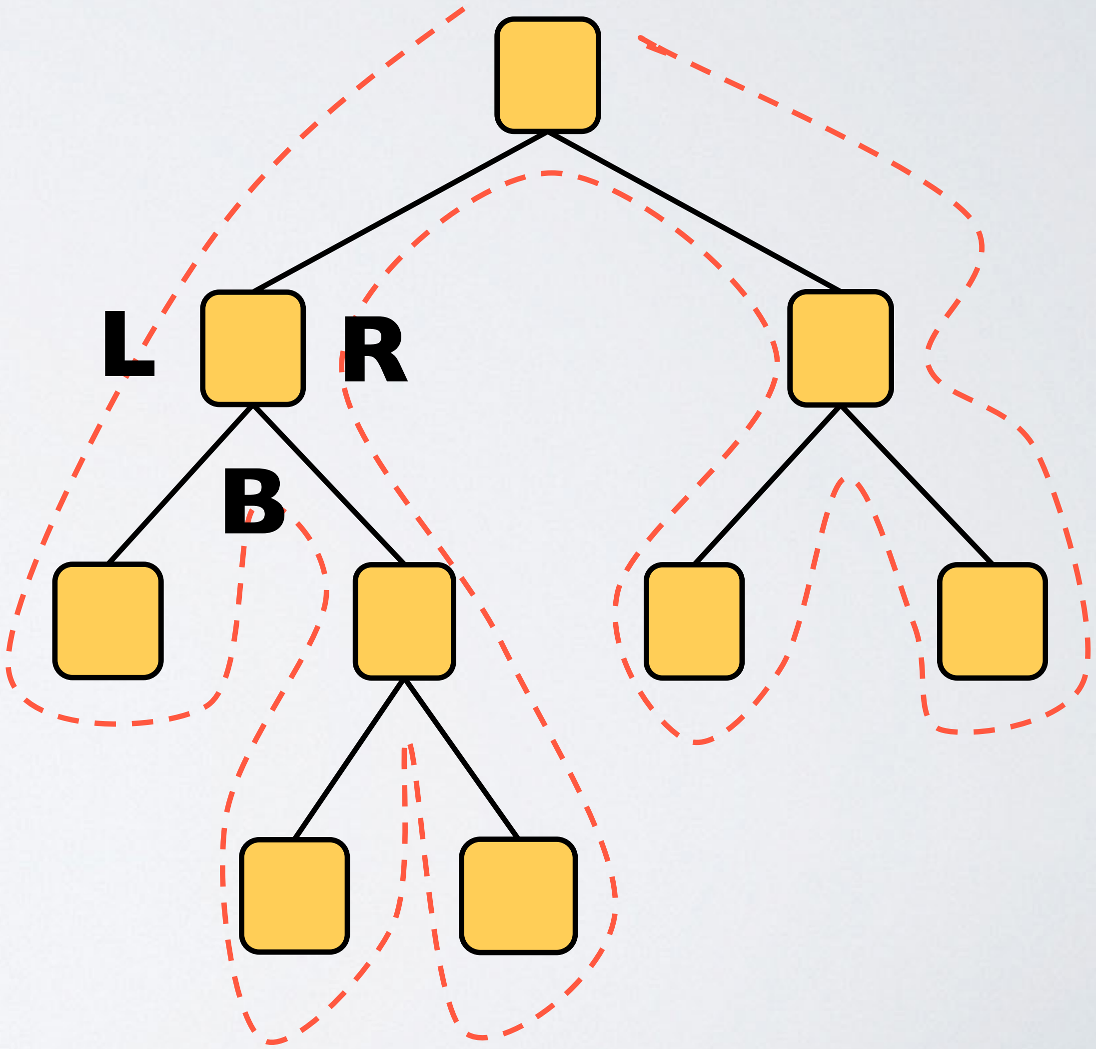
- Generic traversal of binary tree
  - pre-order, post-order and in-order are special cases
- Each node visited 3 times
  - left, bottom, right



Euler Tour Traversal

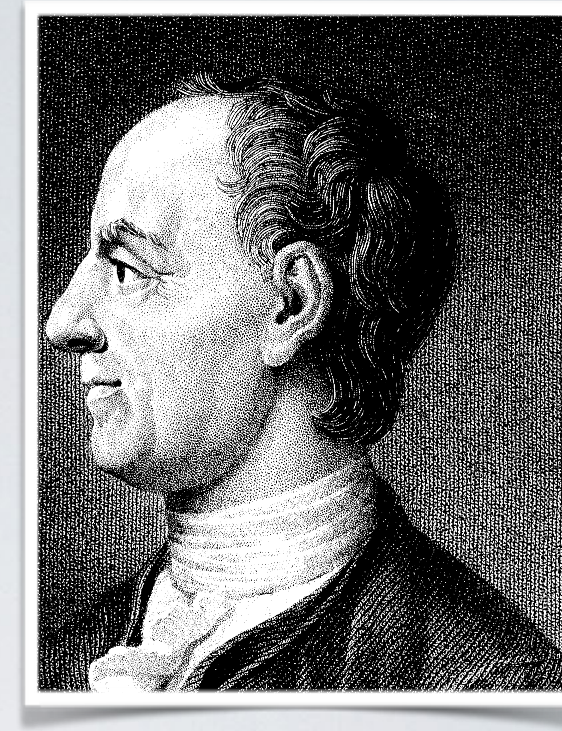


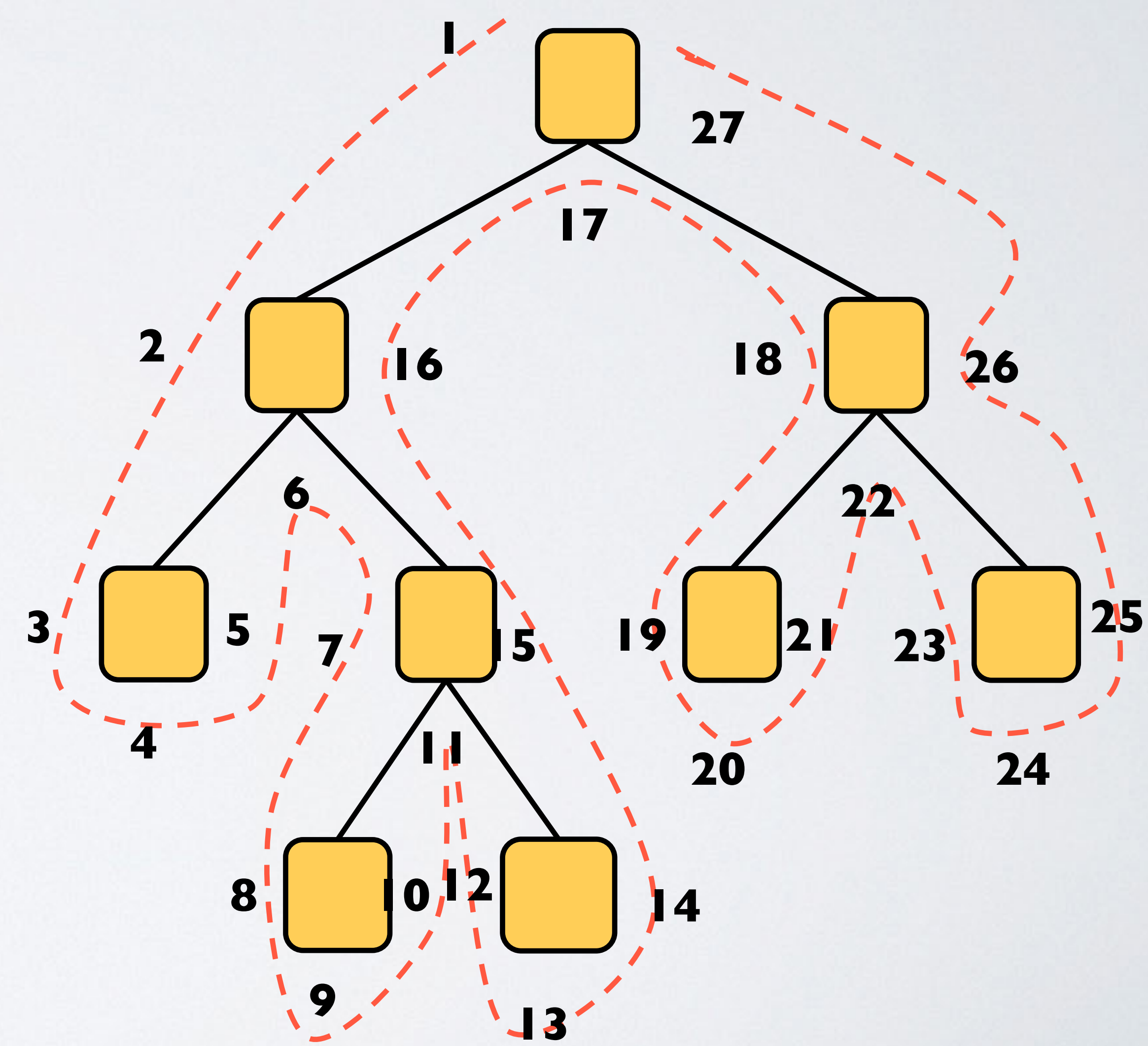
- Visit node on the
  - **left** $\Longrightarrow$ pre-order traversal
  - **bottom** $\Longrightarrow$ in-order traversal
  - **right** $\Longrightarrow$ post-order traversal



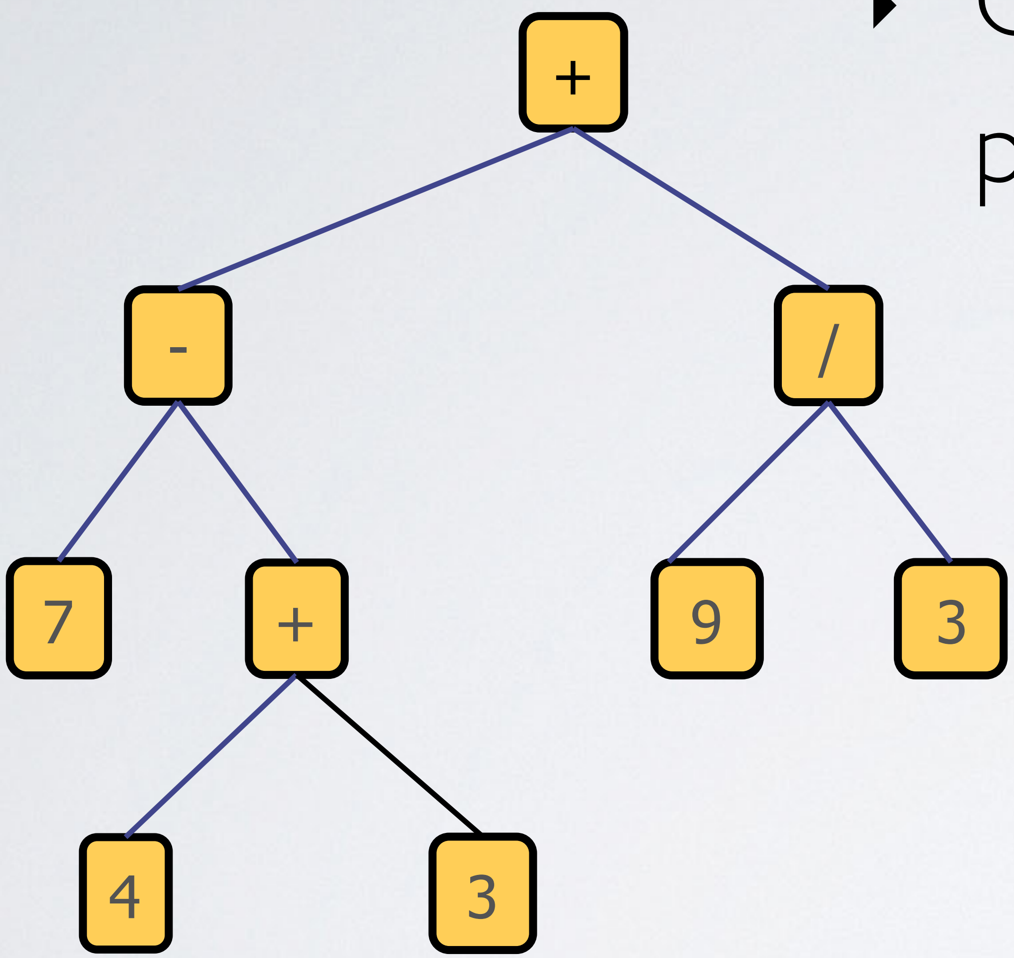
Euler Tour Traversal

```
function eulerTour(node):
  # pre-order
  visitLeft(node)

  if node has left child:
    eulerTour(node.left)

  # in-order
  visitBelow(node)

  if node has right child:
    eulerTour(node.right)

  # post-order
  visitRight(node)
```





Tree Traversal Problems

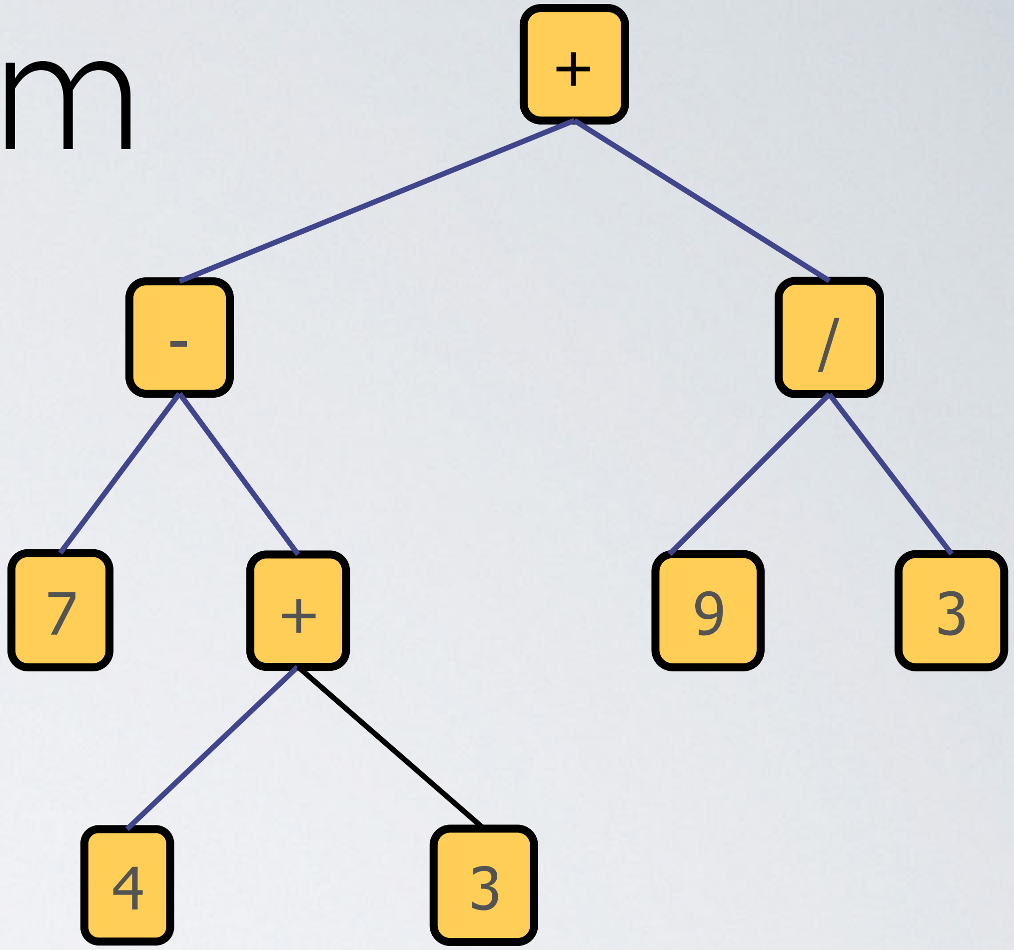


- Given tree, *print out* expression w/ parentheses

$$= (7 - (4 + 3)) + (9 / 3)$$

- Best traversal?
  - **Euler tour**

Tree Traversal Problem



- Best traversal?
  - **Euler tour**
- Internal nodes
  - For pre-order/left visit, print "**(**"
  - For in-order/bottom visit, print operator
  - For post-order/right visit, print "**)**"
- Leaves
  - Don't do anything for pre-order/left and post-order/right visits
  - For in-order/bottom visit, print number

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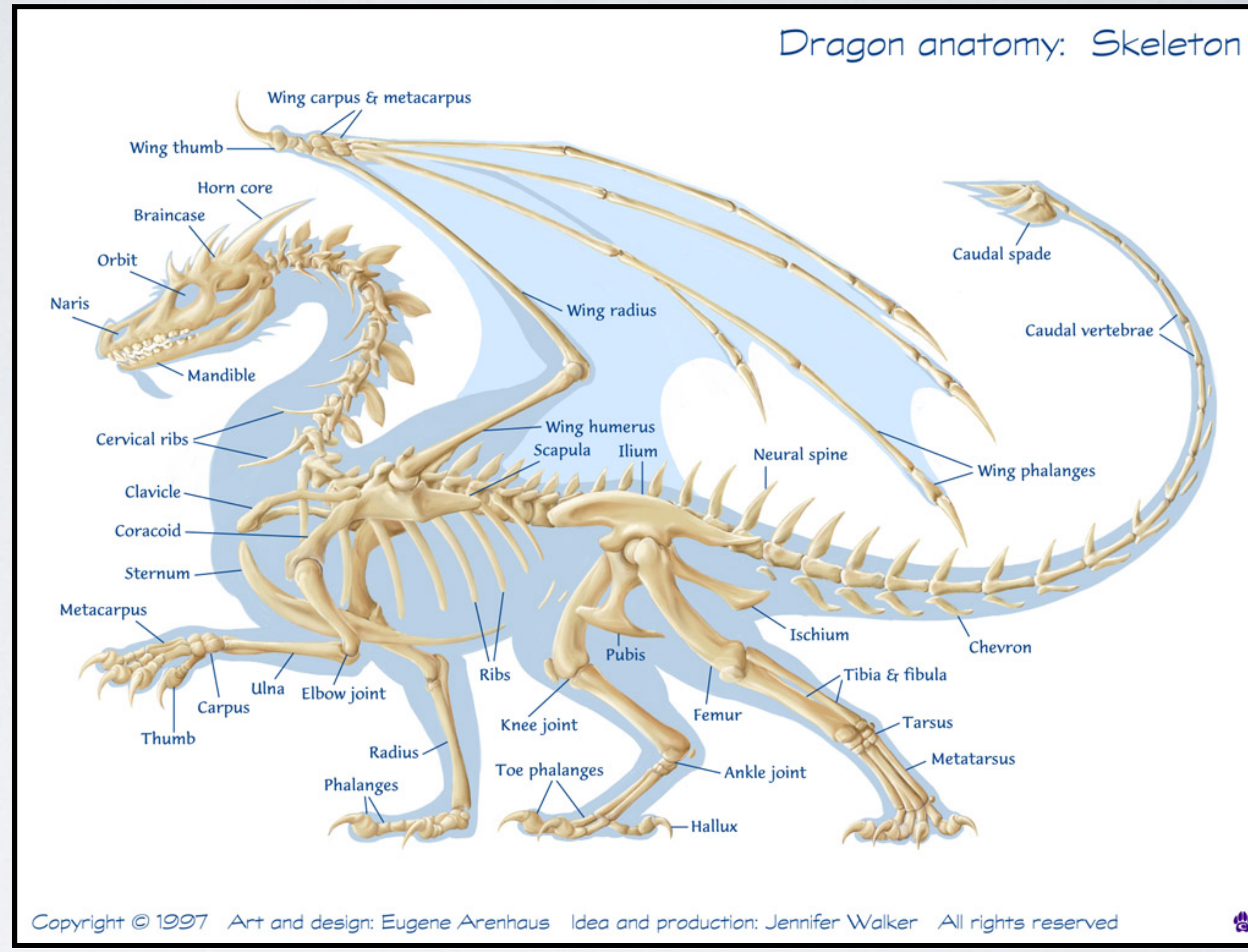
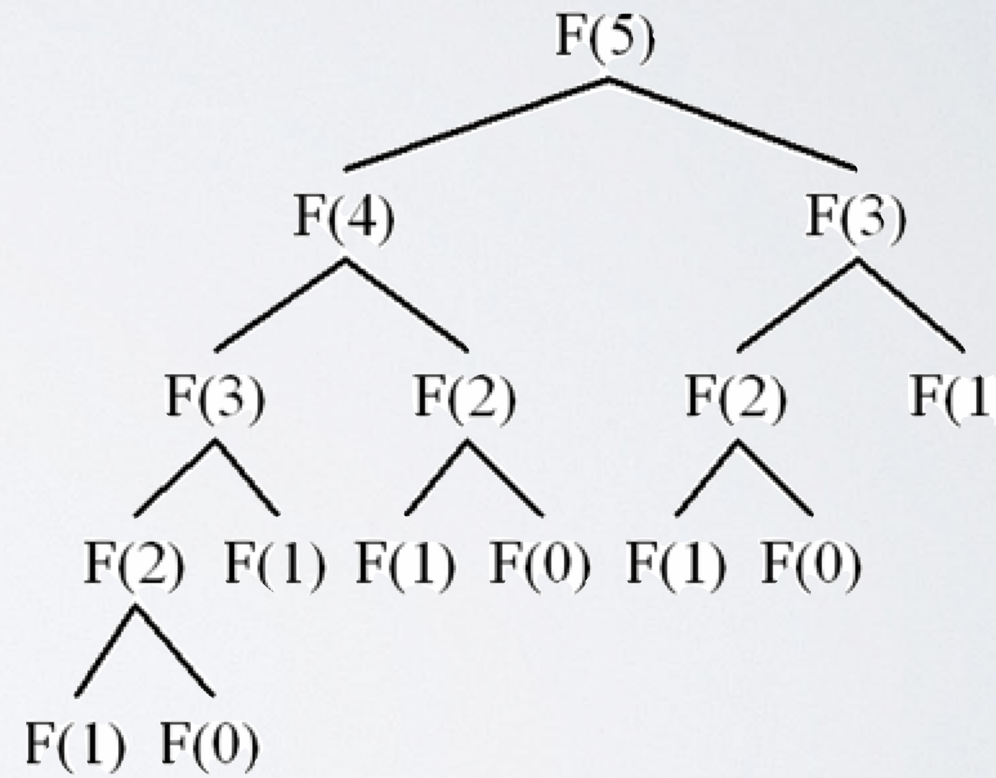
- **Analysis on perfect binary trees**



Analyzing Binary Trees

- Many things can be modeled as binary trees
  - ex: Fibonacci recursive tree

$$F(n) = F(n-1) + F(n-2)$$



Analyzing Binary Trees

- Knowing facts about binary trees can help with *runtime analysis*
  - ex: how many recursive calls are made by a binary recursive tree of height `n`?
- Perfect binary trees are easier to analyze…
  - …so often we use them to estimate analysis of general trees

Analyzing *Perfect* Binary Trees

- Number of nodes in perfect binary tree of height **`h`**:
  - $2^{h+1} - 1$
- Height of a perfect binary tree with **`n`** nodes:
  - $\log_2(n+1)-1$
- Number of leaves in perfect binary tree of height **`h`**:
  - 2^h
- Number of nodes in perfect binary tree with **`L`** leaves:
  - **`2L-1`**

Induction on Perfect Binary Trees

- Can use induction to prove things about PBTs
- Using recursive definition of perfect binary trees
- Tree T is a perfect binary tree if
  - it has only one node
  - has root with left and right subtrees which are both perfect binary trees of same height
  - (if subtrees have height `h`, then T has height `h+1`)

Example Inductive Proof on PBTs

- Prove `P(n)`:
  - number of nodes in a perfect binary tree of height `n` is $f(n)=2^{n+1}-1$
- Base case `P(0)`:
  - number of nodes in perfect binary tree of height `0` is `1` (by definition)
  - $f(0) = 2^{0+1}-1 = 2-1 = 1$
- Inductive hypothesis:
  - assume `P(k)` is true (for some $k \geq 0$)
  - in words: the number of nodes in perfect binary tree of height `k` is $f(k)=2^{k+1}-1$

Example Inductive Proof on PBTs

- Then prove that `P(k+1)` is true:
  - Let `T` be any perfect binary tree of height `k+1`
  - By definition, `T` consists of root with two subtrees, `L` and `R`, which are both perfect binary trees of height `k`
  - By inductive hypothesis, `L` and `R` both have $2^{k+1}-1$ nodes
  - So total number of nodes in `T` is:
    - $2*(2^{k+1}-1)+1 = 2^{k+2}-2+1 = 2^{(k+1)+1}-1$
- Since we've proved
  - `P(0)` is true
  - `P(k)` implies `P(k+1)` (for any `k≥0`)
  - It follows by induction that `P(n)` is true for all `n≥0`

Tree ADT vs. Data Structure

- Is a Tree an ADT or a data structure?
  - It's both
  - The answer depends on the context
- Trees are useful and interesting *abstract* objects
  - that capture parent/child relationships
  - they can be implemented using different data structures
    - some trees can be implemented using arrays
    - they can also be implemented using dictionaries
- But when computer scientists talk about Trees they often mean
  - the "linked tree" data structure
  - trees that are implemented using nodes and pointers