Big-O and Sorting

Lecture 15

Outline

- How do we analyze an algorithm?
- Definition of Big-O notation
- Overview and analysis of sorting algorithms

Importance of Algorithm Analysis (1/2)

- "Performance" of an algorithm refers to how quickly it executes and how much memory it requires
  - performance matters when data grows in size!
  - can observe and analyze performance, then revise algorithm to improve its performance

- Algorithm analysis is so important that all Brown CS students are ***required*** to take at least one course covering it

Importance of Algorithm Analysis (2/2)

- Factors that affect performance
  - computing resources
  - language
  - implementation
  - size of data, denoted *N*
    - number of elements to be sorted
    - number of elements in `ArrayList` to iterate through
    - much faster to search through list of CS15 students than list of Brown students

- This lecture: a brief introduction to **Algorithm Analysis!**
- Goal: to maximize efficiency and conserve resources

Runtime

- **Runtime** of an algorithm varies with the input and typically grows with input size
- In most of computer science we focus on **worst case runtime**
  - easier to analyze and important for unforeseen inputs
- **Average case** is what will happen most often. **Best case** requires least amount of work and is best situation you could have.
  - average case is also important, best case is interesting but not insightful
- How to determine runtime?
  - inspect pseudocode and determine number of statements executed by algorithm as a function of input size
  - allows us to evaluate approximate speed of an algorithm independent of hardware or software environment
  - memory use may be even more important for smaller devices

Elementary Operations

- Algorithmic “time” is measured in **elementary operations**
  - math (+, -, *, /, max, min, log, sin, cos, abs, ...)
  - comparisons (==, >, <=, ...)
  - function (method) **calls** and value **returns** (not counting body of the method)
  - variable assignment
  - variable increment or decrement
  - array **allocation**
  - creating a new object (careful, object's constructor may have elementary ops too!)
- For purpose of algorithm analysis, assume each of these operations takes same time: **“1 operation” –** we are only interested in “asymptotic performance” for large data sets (small differences don’t matter) – see Slide 9

Example: Constant Runtime

```
public int addition(int x, int y) {
      return x+y; //add and return 2 ops
}
```

[1]

- **Always** 2 operations performed
  - 1 addition
  - 1 return
- How many operations performed if this function were to add ten integers? Would it still be constant runtime?

Example: Linear Runtime

```
//find max of a set of positive integers
public int maxElement(int[] a) {
  int max = 0; //assignment, 1 op
  for (int i=0; i<a.length; i++){//3 ops per loop
    if (a[i]> max) { //2 ops per loop
      max = a[i]; //2 ops per loop, sometimes
    }
  }
  return max; //1 op
}
```

Only the largest N expression *without constants* matters!
5n+2, 4n, 300n are all **linear** in runtime.
More about this on following slides!

- Worst case varies proportional to the size of the input list: **7n + 2**
- How many operations if the array had 1,000 elements?
- We'll run the `for` loop more times as the input list grows
- The runtime increase is proportional to N, **linear**

Example: Quadratic Runtime

```
public void printPossibleSums(int[] a) {
  for (i=0; i< a.length; i++) { //2 op per loop
    for (j=0; j<a.length; j++) { //2 op per loop
      System.out.println(a[i] + a[j]); // 4 ops per loop
    }
  }
}
```

- Requires about **$8n^2$** operations (It is okay to approximate!)
- Number of operations executed **grows quadratically!**
- If one element added to list: element must be added with every other element in list
- Notice that linear runtime algorithm on previous slide had only one `for` loop, while this quadratic one has two **nested** `for` loops

Big-O Notation – OrderOf()

- But how to **abstract** from implementation…?
- **Big O** notation
- **O(N)** means an operation is done on each element once
  - N elements * constant operations/element = N operations
- **O(N^2)** means each element is operated on N times
  - N elements * N operations/element = N^2 operations
- Only consider "**asymptotic behavior**" i.e., when N>>1 (N is much greater than 1)
  - N is tiny when compared to N^2

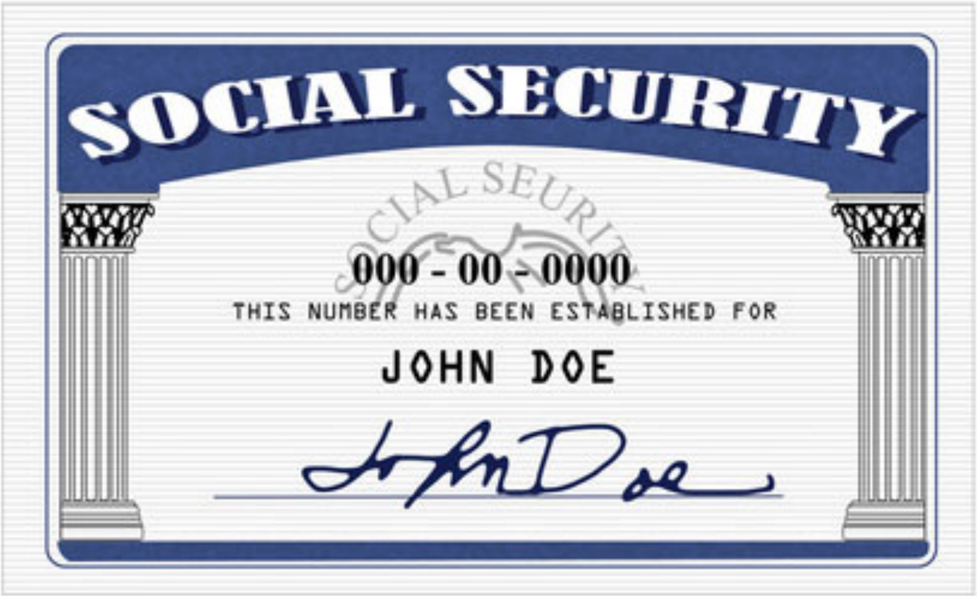
Big-O Constants

- **Important**: Only the largest N expression *without constants* matters.
- We are not concerned about runtime with small numbers of data – we care about running operations on large amounts of inputs
  - $3N^2$ and $500N^2$ are both **$O(N^2)$** – unrealistic if N is small, of course
  - N/2 is **O(N)**
  - $4N^2 + 2N$ is **$O(N^2)$**
- Useful sum that recurs frequently in analysis:

$$1 + 2 + 3 + \cdots + N = \sum_{k=1}^{N} k = N(N+1)/2, \text{ which is } \mathbf{O(N^2)}$$

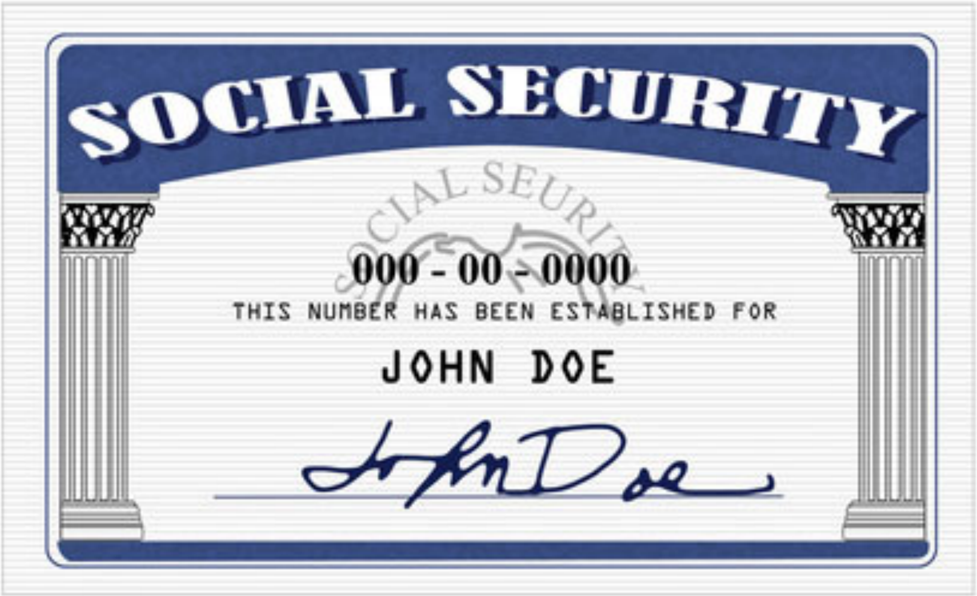
Social Security Database Example (1/3)

- Hundreds of millions of people in the US have a number associated to them
- If 100,000 people named John Smith, each has an individual SSN
- If government wants to look up information they have on John Smith, they use his SSN



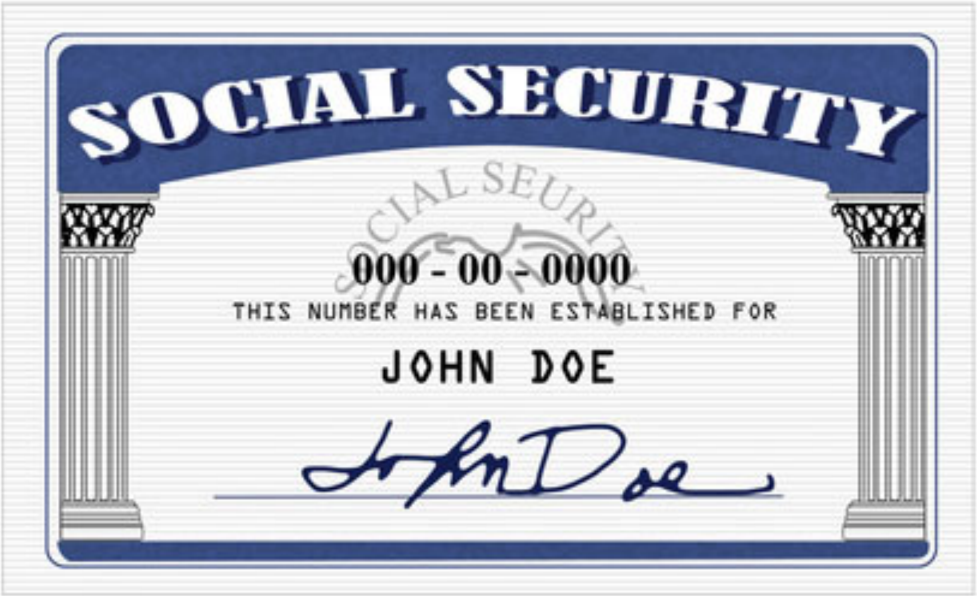
Social Security Database Example (2/3)

- Say it takes 10^{-4} seconds to perform a constant set of operations on one SSN
  - running an algorithm on 5 social security numbers may take 5×10^{-4} seconds, and running an algorithm on 50 will only take 5×10^{-3} seconds
  - both are incredibly fast, a difference in runtime might not be noticeable by an interactive user
  - this changes with large amounts of data, i.e., the actual SS Database



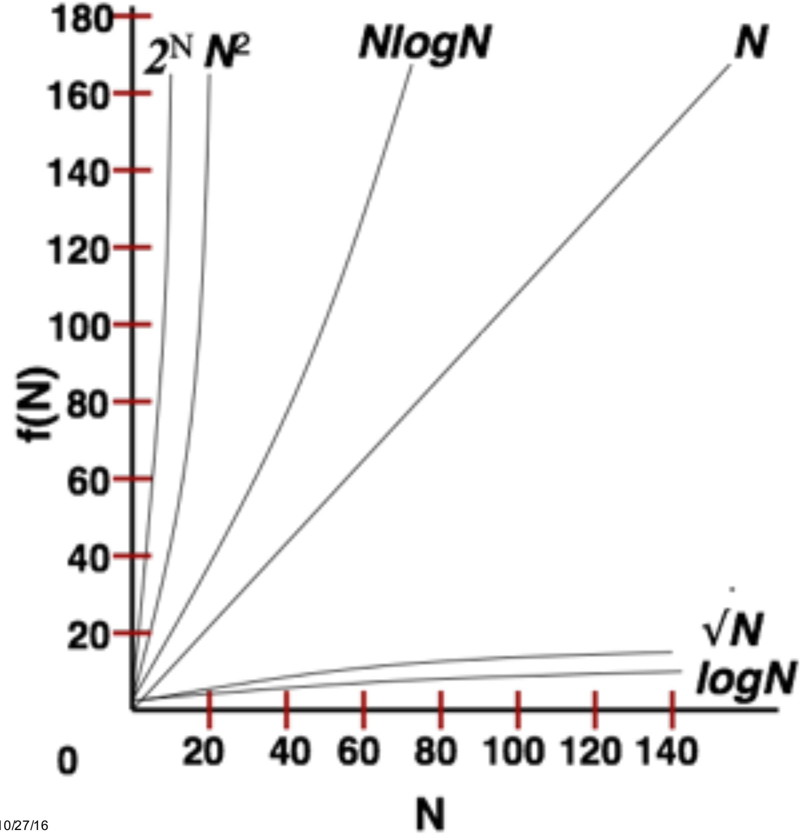
Social Security Database Example (3/3)

- Say it takes 10^{-4} seconds to perform a constant set of operations on one SSN
    - to perform an algorithm with **O(N)** on 300 million people, it will take **8.3 hours**
    - **O(N^2)** takes **285,000 years**
- With large amounts of data, **differences between O(N) and O(N^2) are HUGE!**



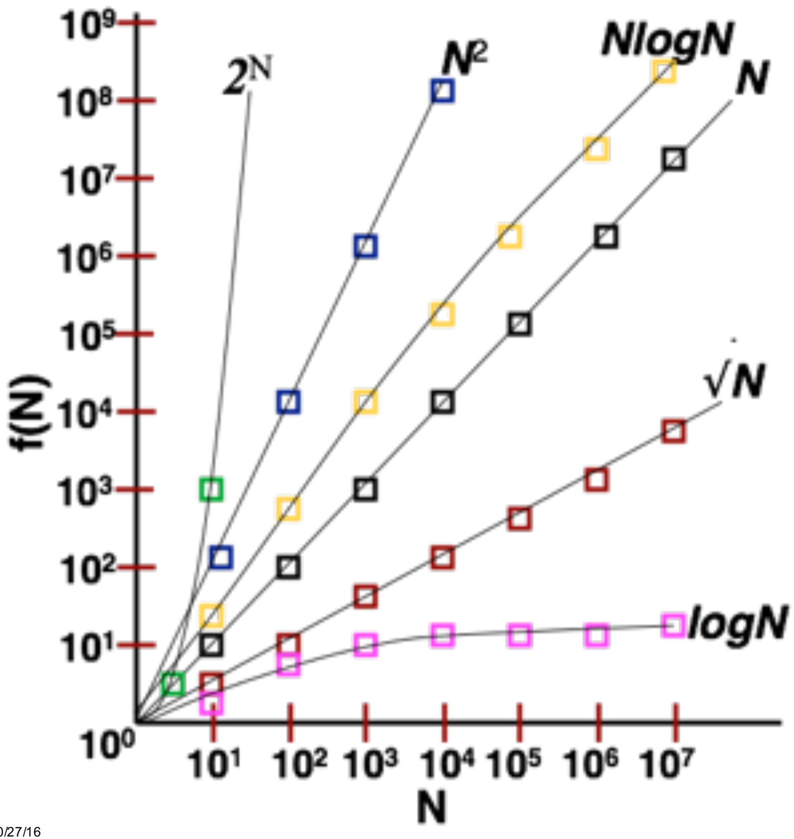
Graphical Perspective (1/2)

- f(N) on a small scale →



Graphical Perspective (2/2)

- f(N) on a larger scale →
- If I have 10 million items ($N = 10^7$)
  - and O(logN) runtime, I perform roughly 7 operations
  - and O(N) runtime, I perform roughly 10 million operations
  - and O(N^2) runtime, I perform roughly 100 trillion operations



Clicker Question (1/3)

What is the big-O **runtime** of this algorithm?

```
public int sumArray(int[] array){
        int sum = 0;
        for (int i=0; i<array.length; i++){
                sum = sum + array[i];
        }
        return sum;
}
```

A) $O(N)$ B) $O(N^2)$ C) $O(1)$ D) $O(2^N)$

Clicker Question (2/3)

What is the big-O **runtime** of this algorithm?

Consider the `getColor` *method in LiteBrite:*

```
public javafx.scene.paint.Color getColor(){
        return _currentColor;
}
```

A) O(N)   B) O(N^2)   C) O(1)   D) O(2^N)

Clicker Question (3/3)

What is the big-O **runtime** of this algorithm?

```
public int sumSquareArray(int dim, int[][] a){
        int sum = 0;
        for (int i=0; i<dim; i++){
                for (j=0; j<dim; j++){
                        sum = sum + a[j][ i];
                }
        }
        return sum;
}
```

A) $O(N)$ B) $O(N^2)$ C) $O(1)$ D) $O(2^N)$

Sorting

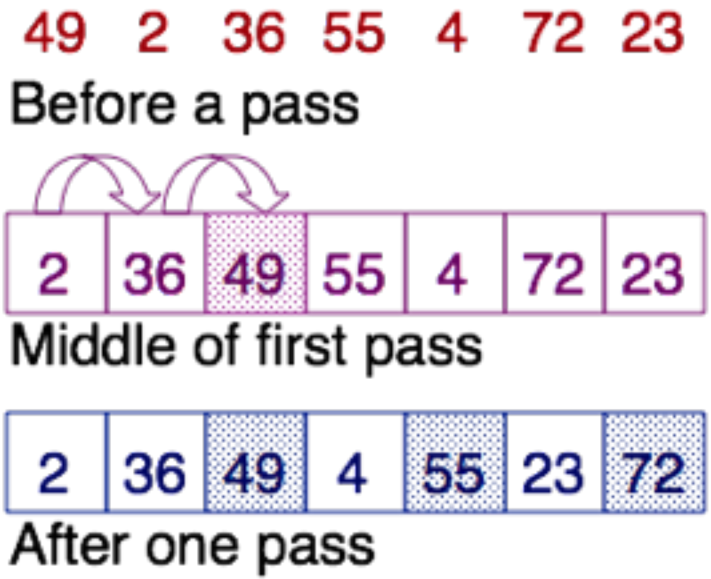
- We use runtime analysis to help choose the best algorithm to solve a problem
- Two common problems: **sorting** and **searching** through a list of objects
- This lecture we will analyze different **sorting** algorithms to find out which is fastest

Sorting – Social Security Numbers

- Consider an example where run-time influences approach
- How would you **sort** every SSN in the Social Security Database in **increasing order?**
- There are multiple known algorithms for sorting a list
    - these algorithms vary in their runtime

Bubble Sort (1/2)

- Iterate through sequence, comparing each element to its right neighbor
- Exchange adjacent elements if necessary; largest element bubbles to the right
- End up with a sorted sub-array on the right. Each time we go through the list, need to switch one fewer item



Bubble Sort (2/2)

- Iterate through sequence, comparing each element to its right neighbor
- Exchange adjacent elements if necessary; largest element bubbles to the right
- End up with a sorted sub-array on the right. Each time we go through the list, need to switch one fewer item
- N is the number of objects in sequence

```
i = N;
sorted = false;
while((i > 1)&&(!sorted))
{
  sorted = true;
  for(int j=1; j<i; j++){
    if (a[j-1] > a[j]) {
      temp = a[j-1];
      a[j-1] = a[j];
      a[j] = temp;
      sorted = false;
    }
  }
  i--;
}
```

Bubble Sort - Runtime

# operations	*Instruction*
1	`i = N;`
1	`sorted = false;`
N	`while((i > 1)&&(!sorted))`
	`{`
(N - 1)	`sorted = true;`
(N-1)+(N-2)+ ... + 2 + 1 = N(N-1)/2	`for(int j=1; j<i; j++){ if (a[j-1] > a[j]) { temp = a[j-1]; a[j-1] = a[j]; a[j] = temp; sorted = false; } }`
(N - 1)	`i--;`
	`}`

```
i = N;
sorted = false;
while((i > 1)&&(!sorted))
{
   sorted = true;
   for(int j=1; j<i; j++){
      if (a[j-1] > a[j]) {
         temp = a[j-1];
         a[j-1] = a[j];
         a[j] = temp;
         sorted = false;
      }
   }
   i--;
}
```

Worst-case analysis (sorted in inverse order):

- the `while`-loop is iterated N-1 times
- iteration i has 2 + 6 (i - 1) operations

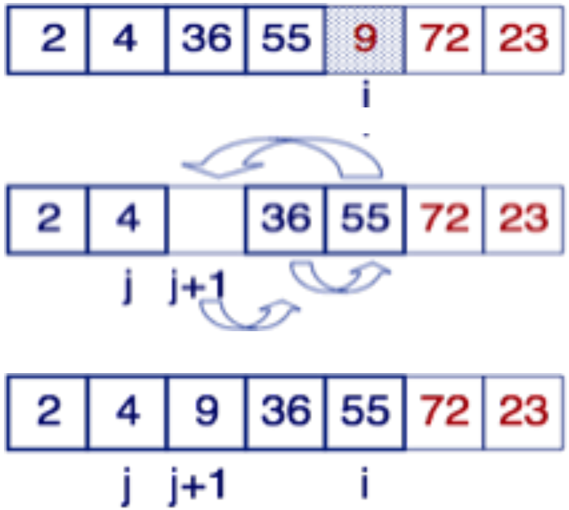
Total:

2+N+2(N-1)+6[(N-1)+…+2+1]=

$3N+6N(N-1)/2 = 3N^2+\ldots = O(N^2)$

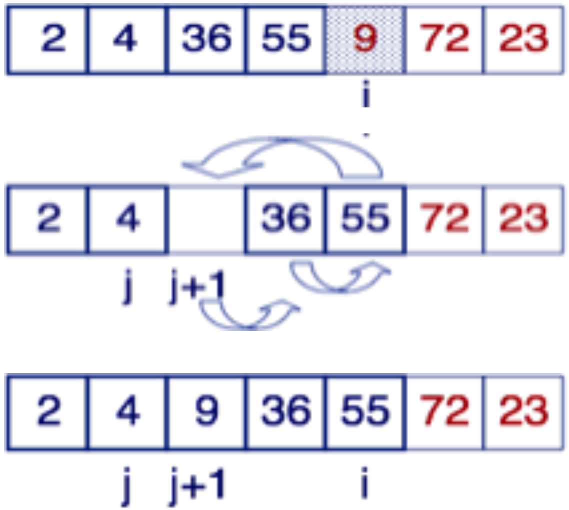
Insertion Sort (1/2)

- Like inserting a new card into a partially sorted hand by bubbling to the left into a sorted subarray; little less brute-force than bubble sort
- Add one element a[i] at a time
- Find proper position, j+1, to the left by shifting to the right a[i-1], a[i-2], ..., a[j+1] left neighbors, until a[j] < a[i]
- Move a[i] into vacated a[j+1]
- **After iteration i<n, the original a[0] ... a[i] are in sorted order, but not necessarily in final position**



Insertion Sort (2/2)

```
for (int i = 1; i < n; i++) {
      int toInsert = a[i];
      int j = i-1;
      while ((j >= 0) && (a[j] > toInsert)) {
             move a[j] forward;
             j--;
      }
      move toInsert to a[j+1];
}
```



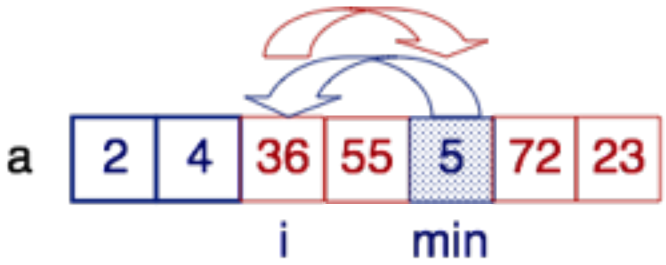
Insertion Sort - Runtime

```
for (int i = 1; i < n; i++) {
      int toInsert = a[i];
      int j = i-1;
      while ((j >= 0) && (a[j] > toInsert)) {
             move a[j] forward;
             j--;
      }
      move toInsert to a[j+1];
 }
```

- `while`-loop inside our `for`-loop. The while loops call on 1, 2, …, N-1 operations… The `for-`loop calls the `while` loop N times.
- $O(N^2)$ because we have to call on a `while` loop with around N operations N different times
- Reminder! **Constants do NOT matter with Big-O**

Selection Sort (1/2)

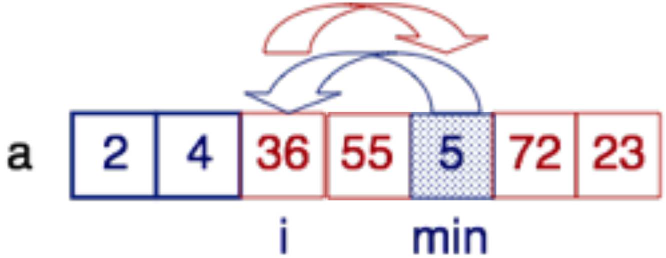
- Find smallest element and put it in a[0]
- Find 2nd smallest element and put it in a[1], etc
- Less data movement (no bubbling)



Selection Sort (2/2)

What we want to happen:

```
for (int i = 0; i < n; i++) {
  find minimum element a[min]
      in subsequence a[i...n-1]
  swap a[min] and a[i]
}
```




```
for (int i = 0; i < n-1; i++) {
  int min = i;
  for (int j = i + 1; j < n; j++) {
    if (a[j] < a[min]) {
      min = j;
    }
  }
  temp = a[min];
  a[min] = a[i];
  a[i] = temp;
}
```

Selection Sort - Runtime

- Most executed instructions are those in inner **for** loop
- Each such instruction is executed (N-1) + (N-2) + … + 2 + 1 times
- Time Complexity: **$O(N^2)$**

```
for (int i = 0; i < n-1; i++) {
  int min = i;
  for (int j = i + 1; j < n; j++) {
     if (a[j] < a[min]) {
        min = j;
     }
  }
  temp = a[min];
  a[min] = a[i];
  a[i] = temp;
}
```

Comparison of Basic Sorting Algorithms

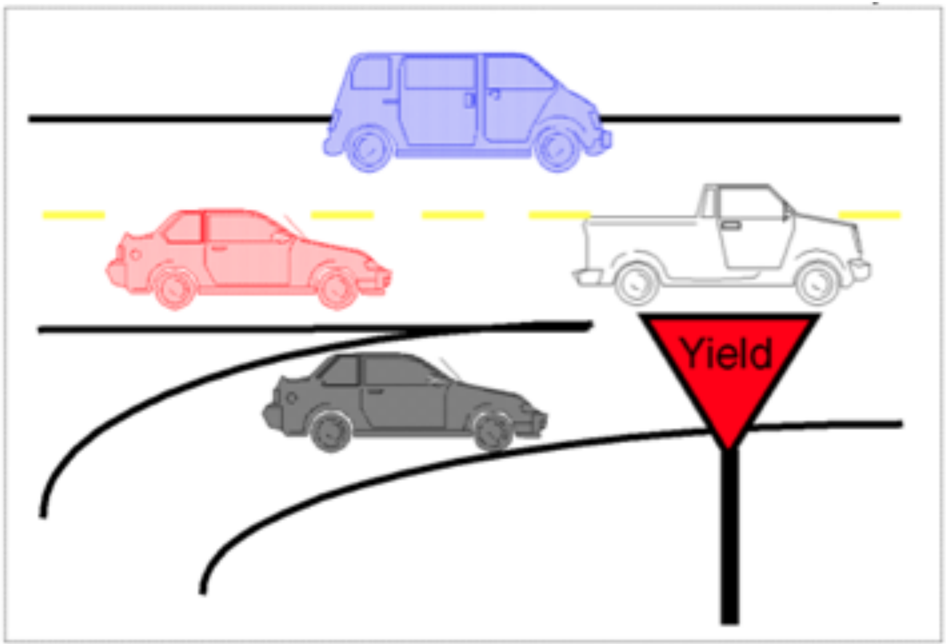
- Differences in **Best** and **Worst** case performance result from the state (ordering) of the input before sorting

- Selection Sort wins on data movement

- For small data, even the worst sort – Bubble – is fine!

		Selection	Insertion	Bubble
Comparisons	Best	$n^2/2$	n	n
	Average	$n^2/2$	$n^2/4$	$n^2/4$
	Worst	$n^2/2$	$n^2/2$	$n^2/2$
Movements	Best	0	0	0
	Average	n	$N^2/4$	$n^2/2$
	Worst	n	$n^2/2$	$n^2/2$

Merge Sort



Recap: Recursion (1/2)

- Recursion is a way of solving problems by breaking them down into smaller sub-problems, and using the results of the sub-problems to find the answer

- Example: You want to determine what row number you're sitting in (in Salomon), but you can only get information from asking the people in front of you
  - they also don't know what row they're in, and ask the people in front of them
  - people in the front row know that they're row 1, since there is no row in front
  - they tell the people behind them, who know that they're 1 behind row 1, so row 2

Recap: Recursion (2/2)

```java
public int findRowNumber(Row myRow) {

    if (myRow.getRowAhead() == null) { // base case!
        return 1;
    } else {
        // recursive case – ask the row in front
        int rowAheadNum = this.findRowNumber(myRow.getRowAhead());

        // my row number is one more than the row ahead’s number
        return rowAheadNum + 1;
    }
}
```

Recursive (Top Down) Merge Sort (1/6)

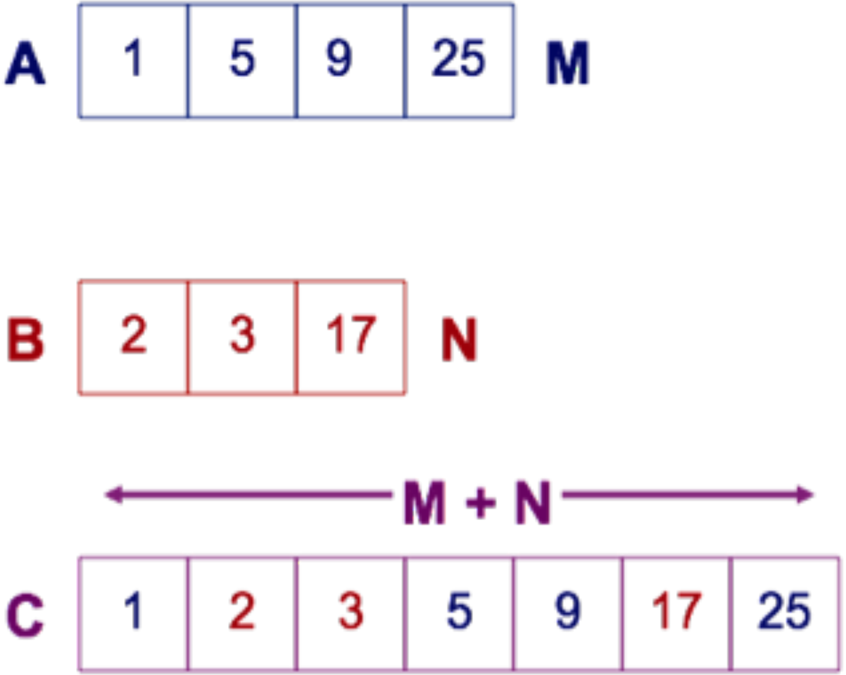
- **Partition** sequence into two sub-sequences of N/2 Elements.

- Recursively **partition** and sort each sub-arrays.

- **Merge** the sorted sub-arrays.

Recursive (Top Down) Merge Sort (2/6)

- **Partition** sequence into two sub-sequences of N/2 Elements.
- Recursively **partition** and sort each sub-arrays.
- **Merge** the sorted sub-arrays.



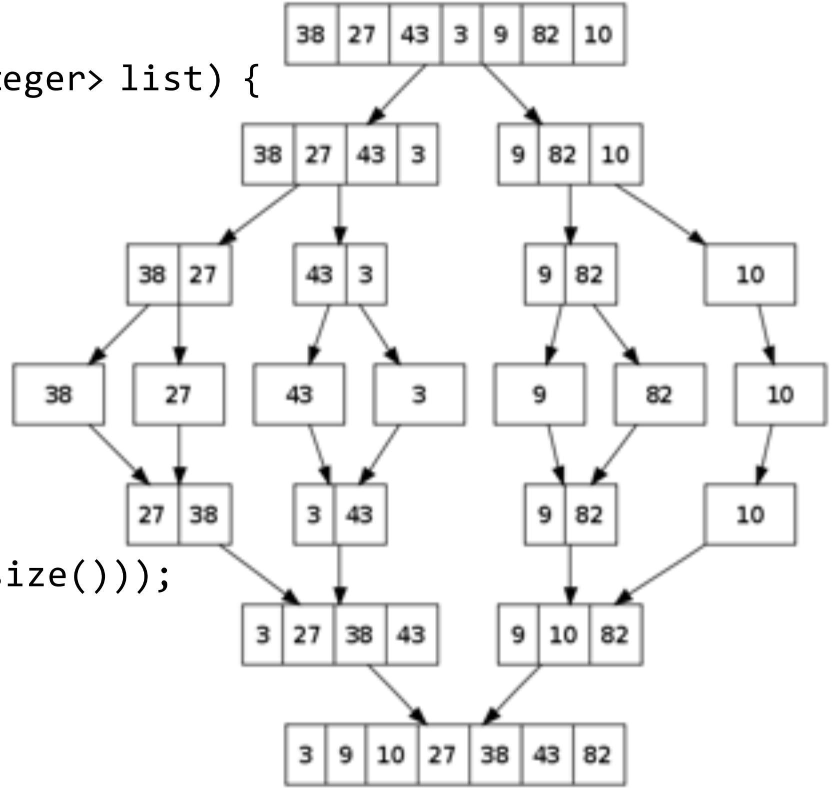
Recursive (Top Down) Merge Sort (3/6)

```
public class Sorts {
  public ArrayList<Integer> mergeSort(ArrayList<Integer> list) {
     if (list.size() == 1) {
       return list;
     }
     int middle = list.size() / 2;
     ArrayList<Integer> left =
          this.mergeSort(list.subList(0, middle));
     ArrayList<Integer> right =
          this.mergeSort(list.subList(middle, list.size()));
     return this.merge(left, right);
  }
  //code for merge() coming next!
}
```

`ArrayList list` is the sequence to sort.

Recursive (Top Down) Merge Sort (4/6)

```
public class Sorts {
  public ArrayList<Integer> mergeSort(ArrayList<Integer> list) {
     if (list.size() == 1) {
       return list;
     }
     int middle = list.size() / 2;
     ArrayList<Integer> left =
          this.mergeSort(list.subList(0, middle));
     ArrayList<Integer> right =
          this.mergeSort(list.subList(middle, list.size()));
     return this.merge(left, right);
  }
  //code for merge() coming next!
}
```



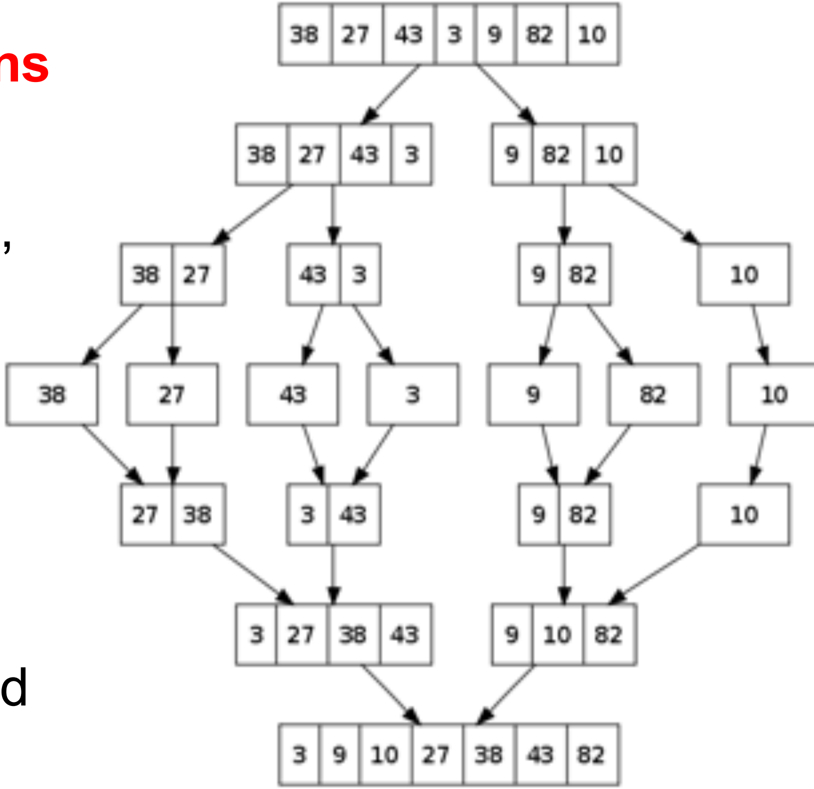
Recursive (Top Down) Merge Sort (5/6)

```
public ArrayList merge(ArrayList<Integer> A, ArrayList<Integer> B) {
   ArrayList<Integer> result = new ArrayList<Integer>();
   int aIndex = 0;
   int bIndex = 0;
   while (aIndex < A.size() && bIndex < B.size()) {
      if (A.get(aIndex) <= B.get(bIndex)) {
         result.add(A.get(aIndex));
         aIndex++;
      } else {
         result.add(B.get((bIndex));
         bIndex++;
      }
   }
   if (aIndex < A.size()) {
      result.addAll(A.subList(aIndex, A.size()));
   }
   if (bIndex < B.size()) {
      result.addAll(B.subList(bIndex, B.size()));
   }
   return result;
}
```

- Add the elements from the two sequences in **increasing order**
- If there are elements left that you haven't added, **add the remaining elements** to your result

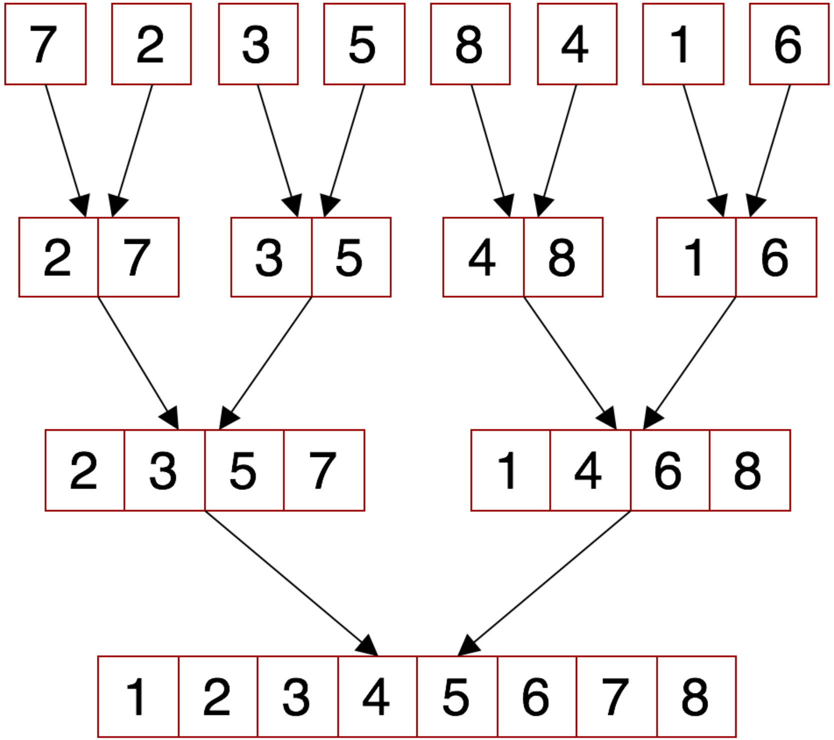
Recursive (Top Down) Merge Sort (6/6)

- Each part of the tree performs **n operations** to **merge** the two subproblems below it.
- Because we divide each sequence by two, the algorithm makes $\log_2 N$ merges.
- ***$O(N\log_2 N)$*** which is way better than ***$O(N^2)$*** we can also drop the log base (2) and say ***$O(N\log N)$***, like how we can remove constants.
- You will learn much more about how to find the runtime of these types of algorithms if you take CS16!



Iterative (Bottom Up) Merge Sort

- Merge sort can also be implemented iteratively…non-recursive!
- Begin by looping through the array of size N, sorting 2 items each. Loop through the array again, combining the 2 sorted items into a sorted item of size 4. Repeat… until there is a single item if size N!
- Number of *iterations* is *log_2N*, rounded up to nearest integer. 1000 elements in the list, *only 10* iterations!!!



Comparing Algorithms Side by Side

Bubble Sort – O(N^2)

Insertion Sort – O(N^2)

Merge Sort - ($N\log_2 N$)

Clicker Question

Which sorting algorithm is the fastest?

A. Bubble Sort

B. Insertion Sort

C. Merge Sort

D. Selection Sort

That’s It!

- Runtime is a very important part of Algorithm analysis!
  - worst case run-time is what we generally focus on
  - know the difference between constant, linear, and quadratic run-time
  - calculate/ define run-time in terms of Big-O Notation
- Sorting!
  - runtime analysis is very significant for Sorting Algorithms!
  - types of sorting algorithms - bubble, insertion, selection, merge sort
  - different algorithms have different performances and time complexities.

What's next?

- You have now seen how different approaches to solving problems can dramatically affect speed of algorithms
  - this lecture utilized arrays to solve most problems

- Subsequent lectures will introduce more ***data structures*** beyond arrays that can be used to handle collections of data

- We can use our newfound knowledge of algorithm analysis to strategically choose different data structures to further speed up algorithms!

Announcements

- Optional Q&A Review sessions in Macmillan 115 tonight at 7:30 and Sunday at noon– focused on recursion
- DoodleJump due dates:

  early: Tuesday 11/1, 11:59pm
  on-time: Thursday 11/3, 11:59pm
  late: Saturday 11/5, 10:00pm

- Start early, start today, start yesterday!