From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored red or black.
- In comparison with its associated (2,4) tree, a red-black tree has:
  - same logarithmic time performance
  - simpler implementation with a single node type

Red-Black Trees

- Root Property: the root is black
- External Property: every leaf is black
- Internal Property: the children of a red node are black
- Depth Property: all the leaves have the same black depth

Height of a Red-Black Tree

- Theorem: A red-black tree storing $n$ entries has height $O(\log n)$
  - Proof:
    - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
    - The search algorithm for a binary search tree is the same as that for a binary search tree
    - By the above theorem, searching in a red-black tree takes $O(\log n)$ time
Insertion

To perform operation $\text{insert}(k, o)$, we execute the insertion algorithm for binary search trees and color red the newly inserted node $z$ unless it is the root.

- We preserve the root, external, and depth properties.
- If the parent $v$ of $z$ is black, we also preserve the internal property and we are done.
- Else ($v$ is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree.

Example where the insertion of 4 causes a double red:

```
  6
  /  \
3     8
 
  6
  /  \
3     8
   / \
 4   z
```

Remedying a Double Red

Consider a double red with child $z$ and parent $v$, and let $w$ be the sibling of $v$.

Case 1: $w$ is black
- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement

Case 2: $w$ is red
- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split

Restructuring

A restructuring remedies a child-parent double red when the parent red node has a black sibling.

- It is equivalent to restoring the correct replacement of a 4-node.
- The internal property is restored and the other properties are preserved.

Restructuring (cont.)

There are four restructuring configurations depending on whether the double red nodes are left or right children.
Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling.
- The parent $v$ and its sibling $w$ become black and the grandparent $u$ becomes red, unless it is the root.
- It is equivalent to performing a split on a 5-node.
- The double red violation may propagate to the grandparent $u$.

\[ \begin{array}{c}
\text{2 4 6 7} \\
\text{4 7}
\end{array} \]

Analysis of Insertion

Recall that a red-black tree has $O(\log n)$ height.

Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes.

Step 2 takes $O(1)$ time.

Step 3 takes $O(\log n)$ time because we perform
- $O(\log n)$ recolorings, each taking $O(1)$ time, and
- at most one restructuring taking $O(1)$ time.

Thus, an insertion in a red-black tree takes $O(\log n)$ time.

Algorithm $\text{insert}(k, o)$

1. We search for key $k$ to locate the insertion node $z$.
2. We add the new entry $(k, o)$ at node $z$ and color $z$ red.
3. while $\text{doubleRed}(z)$
   - if $\text{isBlack}(\text{sibling}(\text{parent}(z)))$
     - $z \leftarrow \text{restructure}(z)$
     - return
   - else if $\text{sibling}(\text{parent}(z))$ is red
     - $z \leftarrow \text{recolor}(z)$

Deletion

To perform operation $\text{remove}(k)$, we first execute the deletion algorithm for binary search trees.

Let $v$ be the internal node removed, $w$ the external node removed, and $r$ the sibling of $w$.

- If either $v$ or $r$ was red, we color $r$ black and we are done.
- Else ($v$ and $r$ were both black) we color $r$ double black, which is a violation of the internal property requiring a reorganization of the tree.

Example where the deletion of 8 causes a double black:

\[ \begin{array}{c}
\text{3 4 6 7} \\
\text{6 7}
\end{array} \]

Remedying a Double Black

The algorithm for remedying a double black node $w$ with sibling $r$ considers three cases:

Case 1: $y$ is black and has a red child
- We perform a restructuring, equivalent to a transfer, and we are done.

Case 2: $y$ is black and its children are both black
- We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation.

Case 3: $y$ is red
- We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies.

Deletion in a red-black tree takes $O(\log n)$ time.
# Red-Black Tree Reorganization

## Insertion

<table>
<thead>
<tr>
<th>Red-black tree action</th>
<th>(2,4) tree action</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>restructuring</td>
<td>change of 4-node</td>
<td>double red removed</td>
</tr>
<tr>
<td>recoloring</td>
<td>split</td>
<td>double red removed or propagated up</td>
</tr>
</tbody>
</table>

## Deletion

<table>
<thead>
<tr>
<th>Red-black tree action</th>
<th>(2,4) tree action</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>restructuring</td>
<td>transfer</td>
<td>double black removed</td>
</tr>
<tr>
<td>recoloring</td>
<td>fusion</td>
<td>double black removed or propagated up</td>
</tr>
<tr>
<td>adjustment</td>
<td>change of 3-node</td>
<td>restructuring or recoloring follows</td>
</tr>
<tr>
<td></td>
<td>representation</td>
<td></td>
</tr>
</tbody>
</table>